Tilings of limit spaces of self-similar groups

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Abstract

We consider tilings of limit spaces of self-similar groups and discuss the following problem: when does the tile of a self-similar group admit a tiling of the limit space under the action of a (self-similar) subgroup?

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1. Introduction. A tiling of a space $\mathcal{X}$ is a collection of compact sets with disjoint interiors whose union is $\mathcal{X}$. The most interesting case is when all sets in a tiling are congruent copies of just one tile. The second part of the Hilbert's eighteenth problem asks whether there exists a polyhedron whose congruent copies can tile $\mathbb{R}^3$ but which is not the fundamental domain of any group. Positive answer to this question inspired intensive research in understanding which compact sets admit a tiling by a group of motions, and which one admit only aperiodic tilings, i.e., tilings with trivial/finite symmetry group.

In this note we consider tilings of limit spaces of self-similar groups. Every contracting self-similar group acts properly and co-compactly on its limit space. We discuss the following question: when does the tile of a group admit a tiling of the limit space by the action of a (self-similar) subgroup (*subgroup tiling*)?

2. Self-similar groups and their limit spaces. Let us recall basic notions from the theory of self-similar groups and their limit spaces developed by V. Nekrashevych [7]. Self-similar group actions are specific actions on the space X^* of all finite words over an alphabet X. A faithful action of a group G on the space X^* is called *self-similar* if for every $g \in G$ and $x \in X$ there exist $h \in H$ and $y \in X$ such that $g(xw) = yh(w)$ for all $w \in X^*$. Iterating we get that for every $g \in G$ and $v \in X^*$ there exist $u \in X^*$ of the same length as v and $h \in G$ such that $g(vw) = uh(w)$ for all $w \in X^*$; the pair (g, v) uniquely determines the element h, which is denoted by $g|_v$.

A self-similar action (G, X^*) is *contracting* if there exists a finite set $\mathcal{N}$ with the property that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words $v \in X^*$ of length $\geq n$. The smallest set $\mathcal{N}$ with this property is called the *nucleus* of the action. Note that

$\mathcal{N} = \mathcal{N}^{-1}$ and $g|_v \in \mathcal{N}$ for all $g \in \mathcal{N}$ and $v \in X^*$. Nucleus can be treated as a finite directed labeled graph with edges $g \xrightarrow{(x,g(x))} g|_x$ for all $g \in \mathcal{N}$ and $x \in X$.

Let (G, X^*) be a contracting action with nucleus $\mathcal{N}$. Consider the space $X^{-\omega} \times G$ of all pairs $(\ldots x_2x_1, g)$, $x_i \in X$ and $g \in G$, with the product topology of discrete sets X and G. The *limit space* $\mathcal{X}_G$ of the action (G, X^*) is defined as the quotient of the space $X^{-\omega} \times G$ by the equivalence relation, where two elements (w_1, g_1) and (w_2, g_2) are equivalent if $g_2g_1^{-1} \in \mathcal{N}$ and the pair (w_1, w_2) is read along a left-infinite path in the nucleus $\mathcal{N}$ ending in the vertex $g_2g_1^{-1}$:

$$(\ldots x_2x_1, g_1) \sim (\ldots y_2y_1, g_2) \quad \Leftrightarrow \quad \ldots \xrightarrow{(x_3|y_3)} h_3 \xrightarrow{(x_2|y_2)} h_2 \xrightarrow{(x_1,y_1)} h_1 = g_2g_1^{-1} \text{ in } \mathcal{N}.$$

The space $\mathcal{X}_G$ is locally compact and Hausdorff. The group G acts properly and co-compactly on $\mathcal{X}_G$ by multiplication from the right: $[(w, g)] \cdot h = [(w, gh)]$.

3. Tiles. The image of $X^{-\omega} \times \{e\}$ in the space $\mathcal{X}_G$ is called the *tile* $\mathcal{T}_G$ of the action. It follows directly from definition that $\mathcal{T}_G$ is compact and covers the limit space under the action of G:

$$\mathcal{X}_G = \bigcup_{g \in G} \mathcal{T}_G \cdot g; \tag{1}$$

the sets $\mathcal{T}_G \cdot g$ are homeomorphic to $\mathcal{T}_G$ and will be called tiles as well. Tiles in this union may have essential intersection, and generally the collection $\{\mathcal{T}_G \cdot g\}_{g \in G}$ is not a tiling of $\mathcal{X}_G$.

Two tiles $\mathcal{T}_G \cdot g_1$ and $\mathcal{T}_G \cdot g_2$ have a nonempty intersection if and only if $g_2g_1^{-1} = \eta \in \mathcal{N}$. Every point from the intersection can be represented by an element from $F_\eta \times \{g_1\}$, where F_η is the set of all sequences $\ldots x_2x_1 \in X^{-\omega}$ that are read from the left labels along a left-infinite path in the nucleus $\mathcal{N}$ ending in the vertex η. The complete preimage of the intersection $\mathcal{T}_G \cdot g_1 \cap \mathcal{T}_G \cdot g_2$ in $X^{-\omega} \times G$ can be described as

$$\bigcup_{\substack{\eta_1, \eta_2 \in \mathcal{N} \\ \eta_1^{-1}g_1 = \eta_2^{-1}g_2}} (F_{\eta_1} \cap F_{\eta_2}) \times \{\eta_1^{-1}g_1\}.$$

Note that the intersection $F_{\eta_1} \cap F_{\eta_2}$ for any $\eta_1, \eta_2 \in \mathcal{N}$ can be described as the set F_v for a vertex v in certain finite labeled directed graph — labeled product of two copies of the nucleus, which can be effectively constructed from $\mathcal{N}$ (see the remark after Corollary 7 in [1]). Therefore it is decidable whether $F_{\eta_1} \cap F_{\eta_2}$ contains an interior point. Note also that the word problem in contracting groups is solvable in polynomial time. Therefore we can distinguish all $\eta_1, \eta_2 \in \mathcal{N}$ that satisfy $\eta_1^{-1}g_1 = \eta_2^{-1}g_2$ from the union above. This gives us a method to check whether two tiles have a common interior point.

A few other observations: If there exists $\eta' \in \mathcal{N}$ such that $\eta'|_v \neq e$ for all $v \in X^*$, then $F_\eta = X^{-\omega}$ for some $\eta = \eta'|_v \in \mathcal{N}$, and the intersection $\mathcal{T}_G \cap \mathcal{T}_G \cdot \eta$ has an interior point. If for every $\eta \in \mathcal{N}$ there exists a word v such that $\eta|_v = e$, then F_η has empty interior for every nontrivial $\eta \in \mathcal{N}$, and different tiles have disjoint interiors. Hence the equality (1) defines a tiling of the limit space if and only if every element of the nucleus is connected with the trivial element by a directed path (equivalently, the tile $\mathcal{T}_G$ has measure one with respect to the uniform measure on the limit space, see [1, Theorem 11]). If the nucleus does not satisfy this condition, then $\mathcal{T}_G$ is covered by the tiles $\mathcal{T}_G \cdot g$ for $g \in \mathcal{N} \setminus \{e\}$ (see [7, Proposition 3.3.7]).

4. Subgroup tilings of limit spaces. I want to raise the following problem.

Problem 1. When does the tile $\mathcal{T}_G$ tile the limit space $\mathcal{X}_G$ with a (self-similar) subgroup $H < G$, i.e., the collection of tiles $\{\mathcal{T}_G \cdot h\}_{h\in H}$ forms a tiling of $\mathcal{X}_G$?

Let us concentrate on the case when the contracting action (G, X^*) is *self-replicating*, i.e., G acts transitively on X and the homomorphism $\phi_G : St_G(x) \to G$, $g \mapsto g|_x$ is surjective. In this case, the nucleus is a generating set of the group, and the limit space $\mathcal{X}_G$ is connected. Then, if there exists a subgroup $H < G$ such that $\{\mathcal{T}_G \cdot h\}_{h\in H}$ is a tiling of $\mathcal{X}_G$, it is finitely generated and has finite index in G. Indeed, let us show that the set $S = \mathcal{N} \cap H = \{h \in H : \mathcal{T}_G \cap \mathcal{T}_G \cdot h \neq \emptyset\} \subset \mathcal{N}$ generates H. Let $H_1 = \langle S\rangle$ and consider sets $\Omega_1 = \mathcal{T}_G \cdot H_1$ and $\Omega_2 = \mathcal{T}_G \cdot (H \setminus H_1)$. Then Ω_1, Ω_2 are closed sets, $\Omega_1 \cup \Omega_2 = \mathcal{X}_G$, and therefore $\Omega_1 \cap \Omega_2 \neq \emptyset$, because the space $\mathcal{X}_G$ is connected. Hence there exist $h_1 \in H_1$ and $h \in H \setminus H_1$ such that $\mathcal{T}_G \cdot h_1 \cap \mathcal{T}_G \cdot h \neq \emptyset$. Then $hh_1^{-1} \in S$ and $h \in H_1$ — contradiction. Hence $H = \langle S\rangle$. Since every pair (w, g) is equivalent to a pair (w', h) for $h \in H$, we get $hg^{-1} \in \mathcal{N}$; hence $G = \mathcal{N} \cdot H$ and H has finite index.

Therefore we can look for such a subgroup H among subgroups generated by subsets $S \subset \mathcal{N}$. One can check whether $\{\mathcal{T}_G \cdot h\}_{h\in H}$ is a tiling of $\mathcal{X}_G$ as follows. Let $\mathcal{N}' \subset \mathcal{N}$ be the set of all $\eta \in \mathcal{N}$ such that $\mathcal{T}_G \cap \mathcal{T}_G \cdot \eta$ has empty interior, which can be found as shown above. Given $S \subset \mathcal{N}$ compute $S' = \langle S\rangle \cap \mathcal{N}$. Then $\mathcal{T}_G \cdot h_1 \cap \mathcal{T}_G \cdot h_2$ has empty interior for all $h_1, h_2 \in H = \langle S\rangle$, $h_1 \neq h_2$, if and only if $S' \subset \mathcal{N}'$. It is left to check that $\{\mathcal{T}_G \cdot h\}_{h\in H}$ covers $\mathcal{X}_G$, which is equivalent to $\mathcal{T}_G \cdot \mathcal{N} \subset \mathcal{T}_G \cdot H$, because $\mathcal{T}_G \cdot G = (\mathcal{T}_G \cdot \mathcal{N}) \cdot H$. Note that if $\mathcal{T}_G \cdot h \cap \mathcal{T}_G \cdot \eta \neq \emptyset$ for $\eta \in \mathcal{N}$ then $h \in \mathcal{N} \cdot \mathcal{N}$. Then the tile $\mathcal{T}_G \cdot \eta$ is covered by the tiles $\mathcal{T}_G \cdot h$ for $h \in H$ if

$$\bigcup_{\substack{h\eta^{-1} \in \mathcal{N} \\ h \in H \cap \mathcal{N} \cdot \mathcal{N}}} F_{h\eta^{-1}} = X^{-\omega},$$

which is decidable (if we know $H \cap \mathcal{N} \cdot \mathcal{N}$).

Concerning tiling with a self-similar subgroup, the situation is more clear. As discussed above, if $g|_v \neq e$ for all $v \in X^*$, then $\mathcal{T}_G \cap \mathcal{T}_G \cdot g|_v$ contains an interior point for some word v; therefore such elements g cannot belong to a self-similar subgroup H that yields tiling of $\mathcal{X}_G$. Let $H \subset G$ be the subset of all elements $g \in G$ such that for any word $v \in X^*$ there exists a word $u \in X^*$ such that $g|_{vu} = 1$. Now every self-similar subgroup that gives a tiling of the limit space is contained in H. But H itself is a self-similar subgroup. Indeed, for any $g, h \in H$ and $v \in X^*$ there exist $u, w \in X^*$ such that $h|_{vu} = e$ and $g|_{h(vuw)} = e$; then $gh|_{vuw} = g|_{h(vuw)}h|_{vuw} = e$ and $gh \in H$. Since tiles $\mathcal{T}_G \cdot g$ for $g \in H$ have disjoint interiors, it is left to check that they cover the limit space, what can be done as above.

5. Tilings of limit spaces of abelian and nilpotent groups. We have discussed how to answer Problem 1 for a given contracting group. The question can be asked in a different form: for a given group or class of groups, is it true that for every contracting action the tile of the action tiles the limit space with a (self-similar) subgroup? Maybe an interesting result can be obtained if we fix a group and its virtual endomorphism ϕ_G. For example, for every contracting action given by a bounded automaton the covering (1) is a tiling of $\mathcal{X}_G$, in other words, the answer is positive for post-critically finite limit spaces.

The question is difficult even in abelian case. Every contracting self-replicating action of the free abelian group $\mathbb{Z}^n$ can be described as follows (see [7, Section 6.2]). Let $A \in M_n(\mathbb{Z})$ be an integer matrix with all eigenvalues greater than one in modulus. The lattice $\mathbb{Z}^n$ is invariant under A; let $D = \{d_x, x \in X\}$ be a set of coset representatives for $A(\mathbb{Z}^n)$ in $\mathbb{Z}^n$ (here X is just an index set treated as alphabet, $|X| = |D| = |\det(A)|$). For every $x \in X$ and $v \in X^*$ the action of an element $a \in \mathbb{Z}^n$ is defined recursively by the rule

$$a(xv) = yb(v) \quad \text{and} \quad b = a|_x := A^{-1}(a + d_x - d_y),$$

where $y \in X$ is uniquely defined by the condition $a + d_x - d_y \in A(\mathbb{Z}^n)$. This action $(\mathbb{Z}^n, X^*)$ is self-similar, contracting, and self-replicating. The limit space $\mathcal{X}_G$ of the action is homeomorphic and $\mathbb{Z}^n$-equivariant to $\mathbb{R}^n$ with the natural action of $\mathbb{Z}^n$ by shifts. Under the homeomorphism, the tile $\mathcal{T}_G$ is mapped to the set

$$T(A, D) = \left\{ \sum_{k=1}^{\infty} A^{-k} d_k : d_k \in D \right\} \subset \mathbb{R}^n,$$

which is known as the *self-affine tile* associated with the pair (A, D). Tiling properties of self-affine tiles were considered in [4], motivated by construction of orthonormal wavelet bases in $\mathbb{R}^n$, and it was conjectured that every self-affine tile gives a lattice tiling of $\mathbb{R}^n$ with some lattice $\Gamma < \mathbb{Z}^n$. The conjecture was confirmed in [5] and [2]; the proof is complicated and was based on a partial case of Hyperplane Zeros Conjecture (completely proved later in [6]) about the zero set of real analytic periodic functions. This result is my initial motivation for posing Problem 1 — I think that self-similar actions are the right settings for natural generalization. The analysis we made in the previous section basically generalizes the main result from [3], which gives a method for determining all possible lattice tilings for a given self-affine tile.

Interestingly, not every self-affine tile gives a tiling with a self-similar lattice. Consider the example from [5, page 85]:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \qquad D = \{(0,0), (3,0), (0,1), (3,1)\}.$$

The associated self-similar action of $\mathbb{Z}^2 = \langle a = (1,0), b = (0,1) \rangle$ over the alphabet $X = \{1, 2, 3, 4\}$ is given by

$$\begin{array}{llll} a(1v) = 2a^{-1}(v) & a(2v) = 1a^2(v) & a(3v) = 4a^{-1}(v) & a(4v) = 3a^2(v) \\ b(1v) = 3v & b(2v) = 4v & b(3v) = 2(a^{-2}b)(v) & b(4v) = 1(ab)(v) \end{array}$$

(we use multiplicative notations). The nucleus of the action is

$$\mathcal{N} = \{(0,0), (\pm 1, 0), (0, \pm 1), (\pm 2, 0), (\pm 3, 0), \pm(\pm 1, 1), \pm(\pm 2, 1), \pm(-3, 1), \pm(-4, 1)\}.$$

The elements $\eta \in \mathcal{N}$ with the property that for every $v \in X^*$ there exists $u \in X^*$ such that $\eta|_{vu} = e$ constitute the set $\mathcal{N}_0 = \{(0,0), (\pm 3, 0)\}$. The space $\mathbb{R}^2$ is not covered by the self-affine tile $T(A, D)$ under the action of $\langle \mathcal{N}_0 \rangle$. Therefore $T(A, D)$ does not tile $\mathbb{R}^2$ with any self-similar lattice as we discussed in the previous section. Nevertheless, $T(A, D)$ possesses a lattice tiling for the lattice $\langle (3, 0), (0, 1) \rangle$.

It is interesting how far can be generalized the result about lattice tilings of self-affine tiles. I think one can hope for a positive result dealing with nilpotent groups.

Conjecture 1 (Lattice tiling conjecture for nilpotent groups)**.** Let (G, X) be a contracting self-replicating self-similar action of a finitely generated torsion-free nilpotent group G. There exists a subgroup $H < G$ such that the collection of tiles $\{\mathcal{T}_G \cdot h\}_{h \in H}$ forms a tiling of the limit space $\mathcal{X}_G$.

Under the conditions of the conjecture, the limit space $\mathcal{X}_G$ is homeomorphic to the unique real nilpotent, connected and simply connected Lie group L, Mal'cev completion of G. Directly following the path laid in [5, 2] requires a generalization of certain results from functional analysis to nilpotent Lie groups.

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