CHAPTER 12

# Open Problems

This chapter collects questions for which no answer is yet known. Here we use the geometric definition of branch group given in Definition 1.13, except in Question 2.

The theory of branch group is a recent development and the questions arise and die almost every day, so there are no longstanding problems in the area and there is no easy way to say which questions are difficult and which are not. The list below just serves as a list of problems that one should naturally ask at this given moment. We kindly invite the interested readers to solve as many of the problems below.

**Added in the proof.** The authors had a chance to proofread the article 18 months after the initial submission. Some of the proposed problems have been solved in the meantime and we include appropriate comments to that effect.

QUESTION 1. *Is there a finitely generated fractal regular branch group $G$, branched over $K$, acting on the binary tree, such that the index of the geometric embedding of $K \times K$ into $K$ is two?*

QUESTION 2. *Every branch group from Definition 1.1 acts canonically on the tree determined by its branch structure as a group of tree automorphisms. Is the kernel of this action necessarily central?*

QUESTION 3. *Does every finitely generated branch $p$-group, where $p$ is a prime, satisfy the congruence subgroup property?*

QUESTION 4. *Is every finitely generated branch group isomorphic to a spinal group?*

QUESTION 5. *Is the conjugacy problem solvable in all branch groups with solvable word problem?*

QUESTION 6. *When do the defining triples $\omega$ and $\omega'$ define non-isomorphic examples of branch groups in* [**Gri84, Gri85a**] *and more generally in $\mathsf{G}$ and $\mathsf{GGS}$ groups?*

QUESTION 7. *Which branch groups have finite $L$-presentations? Finite ascending $L$-presentations? In particular, what is the status of the Gupta-Sidki 3-group?*

All that is known at present is that the Gupta-Sidki 3-group has a finite endomorphic presentation.

QUESTION 8. *Do there exist finitely presented branch groups?*

QUESTION 9. *Is it correct that there are no finitely generated hereditary just-infinite torsion groups?*

QUESTION 10. *Is every finitely generated just-infinite group of intermediate growth necessarily a branch group?*

QUESTION 11. *Is every finitely generated hereditarily just-infinite group necessarily linear?*

QUESTION 12. *Recall that $G$ has bounded generation if there exist elements $g_1 \dots, g_k$ in $G$ such that every element in $G$ can be written as $g_1^{n_1} \dots g_k^{n_k}$ for some $n_1, \dots, n_k \in \mathbb{Z}$. Can a just-infinite branch group have bounded generation? Can infinite simple group have bounded generation?*

QUESTION 13. *What is the height (in the sense of* [**Pri80**]*, see Chapter 5) of a Grigorchuk 2-group $G_\omega$ when the defining sequence $\overline{\omega}$ is not periodic? Same question for arbitrary $G$ groups. In particular, can the height be infinite?*

QUESTION 14. *Is every maximal subgroup in a finitely generated branch group necessarily of finite index?*

QUESTION 15. *Is there is a finitely generated branch group containing the free group $F_2$ on two generators?*

Positive answer is provided by Said Sidki and John Wilson in [**SW02**]

QUESTION 16. *Are there finitely generated branch groups with exponential growth that do not contain the free group $F_2$?*

QUESTION 17. *Is there a finitely generated branch group whose degree of growth is $e^{\sqrt{n}}$? Is there such a group in the whole class of finitely generated groups?*

QUESTION 18. *What is the exact degree of growth of any of the basic examples of regular branch groups (for example $\mathfrak{G}$, $\Gamma$, $\overline{\Gamma}$, ...)?*

QUESTION 19. *What is the growth of the Brunner-Sidki-Vieira group (see* [**BSV99**] *and Section 4.3)?*

It is known that the Brunner-Sidki-Vieira group does not contain any non-abelian free groups, but it is not known whether it contains a non-abelian free monoid. Note that this group is not a branch group, but it is a weakly branch group.

QUESTION 20. *Are there finitely generated non-amenable branch groups not containing the free group $F_2$?*

QUESTION 21. *Is it correct that in each finitely generated fractal branch group $G$ every finitely generated subgroup is either finite or Pride equivalent with $G$?*

A stronger property holds for $\mathfrak{G}$, namely, John Wilson and the second author have proved in [**GW01**] that every finitely generated subgroup of $\mathfrak{G}$ is either finite or commensurable with $\mathfrak{G}$. Claas Röver has announced that the answer is also positive for the Gupta-Sidki group $\overline{\overline{\Gamma}}$. However, the answer is negative in general.

QUESTION 22. *Do there exist branch groups with the property (T)?*