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Automata, Dynamical Systems, and Groups

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Abstract—This paper is devoted to the groups of finite automata and their applications in algebra, dynamical systems, and geometry. The groups of synchronous automata as well as the groups of asynchronous automata are considered. The problems of reduction of finite asynchronous automata, the types of growth of finite synchronous automata, and the conditions of embeddability of groups in the group of automata are studied. The automorphism groups of cellular automata are investigated. A group of rational homeomorphisms of the Cantor set is introduced. The dynamics, on the boundary of a tree, determined by an automaton group is investigated. Certain unsolved problems are formulated.

1. INTRODUCTION

Automata play a decisive role in the field of science that is now called informatics (or computer science in the West). The most important functions of automata are the recognition and transformation of sets. Accordingly, automata are classified into two types: acceptors and transducers.

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Both types of automata play an important role in algebra and the theory of dynamical systems. There exist a considerable amount of monographs and survey papers [65, 18, 17, 35, 39, 76, 69, 52, 55, 36] (not to mention research works) devoted to the application of automata in the aforementioned fields of mathematics.

The aim of this paper (which combines the character of survey and research) is to represent nontraditional applications of automata in algebra, theory of dynamical systems, theory of graphs, and spectral theory.

We mainly deal with transducers (both synchronous and asynchronous). If the input and output alphabets of an automaton coincide and the automaton is initial (i.e., has an initial state), then it induces a transformation of a space of sequences into itself. These sequences may be either finite or infinite. In the latter case, we have a continuous transformation of a Cantor set, which is naturally identified with the space of sequences. Conversely, any continuous transformation is defined by a certain automaton (in general, with an infinite number of states). An important class of transformations of a Cantor set is represented by the homeomorphisms defined by finite automata, which are referred to as rational homeomorphisms. Examples of rational homeomorphisms are given by the adding machines and the Bernoulli shifts.

The compositions of mappings defined by automata correspond to the operation of composition of automata that transforms the set of equivalence classes of automata into a semigroup. The groups of invertible automata play an important role; among these groups, we highlight the groups of finite automata.

The groups of synchronous automata depend on the cardinality of the alphabet and are residually finite. At the same time, the dependence on the alphabet and the property of residual finiteness vanish when passing to asynchronous automata.

One of the wonderful phenomena in modern mathematics is the discovery of the fact that even the simplest automata with the number of states $2, 3, 4, \dots$ generate most complicated groups that possess rare and extraordinary properties.

It turned out that the groups of finite automata give an answer to Burnside's question about the existence of infinite finitely generated periodic groups. For the first time, this problem was solved by Golod with the use of the Golod–Shafarevich theorem [70]. However, the simplest and most elegant solution was obtained by means of automata [63, 71]; the application of automata to Burnside's problem was suggested by Glushkov [69, p. 46].

The second discovery consisted in the following. Among the groups of automata, there exist those of intermediate growth between exponential and polynomial; this discovery gave the solution to the Milnor problem [72]. The concept of growth can be defined not only for groups and semigroups but also for finite automata; the growth of an automaton coincides with the growth of the semigroup defined by this automaton [74]. At present, there are examples of two-state automata with the intermediate growth.

Automata are directly related to branch groups, i.e., the groups that act on spherically homogeneous rooted trees whose lattice of subnormal subgroups has a structure similar to that of a tree [24]. The importance of this class of groups is due to the fact that they constitute one of three classes into which the class of just infinite groups (i.e., infinite groups with finite proper quotients) is naturally decomposed.

One of key problems in the theory of groups of finite automata is the problem of embeddability of other known classes of groups into these groups. This problem is solved positively for free groups, free abelian groups, certain classes of linear and solvable groups, and R. Thompson's groups. It also turned out that the groups of shift automorphisms of finite type (i.e., the groups of invertible cellular automata) are the groups of finite (asynchronous) automata.

Any group G generated by synchronous automata acts on a k-regular infinite rooted tree T, where k is the cardinality of the alphabet, and, hence, acts by homeomorphisms also on the boundary ∂T of the tree. A group generated by asynchronous automata also acts by homeomorphisms on ∂T; however, this action is not induced by the action of the tree automorphisms. In the synchronous case, the action on the boundary keeps invariant the uniform measure μ on the boundary. Thus, we obtain a topological dynamical system $(G, \partial T)$ and a metric dynamical system $(G, \partial T, \mu)$. Note that, actually, we deal with the dynamical systems on the Cantor set, since the boundary ∂T belongs to the topological type of this set.

Recently, there have been a rapid progress in the investigation of topological dynamics on the Cantor set [22, 23, 37], and the present work makes a certain contribution to this field of research. One of typical problems is the problem concerning the properties of partitioning into orbits, where the so-called confinal partition plays an important role. Usually, the problems of amenability (of the group G and the action of G on ∂T), the Kazhdan T-property, and many other problems of the asymptotic group theory play an important role in the trajectory theory.

To a finitely generated group of automata that acts spherically transitively, there corresponds an infinite regular graph on the boundary, as well as a sequence of finite regular graphs that approximate the infinite graph. These graphs are isomorphic to the Schreier graphs $\Gamma(G, P, S)$ and $\Gamma(G, P_n, S)$, where $P, P_n < G$ are parabolic subgroups (the stabilizers of a point of the boundary ∂T and of the vertex of the nth level of the tree T, respectively) and S is a system of generators of the group. They posses a number of interesting properties. For example, in many cases, infinite graphs associated with the branch groups of automata are substitutional graphs of fractal type and have a polynomial growth [7, 6]. One may hope that, using the automata that generate a free group or a group close to a free one, one can obtain new series of expanders and even the Ramanujan graphs (see [41] for the definitions of these concepts).

A finite automaton can also be associated with the concept of spectrum, and this can be done in two ways. The spectrum can be defined, first, as the spectrum of a (noncommutative) dynamical system generated by this automaton and, second, as the spectrum of a discrete Laplace operator on the graph $\Gamma(G, P, S)$. If the parabolic group P is amenable or the action of G on G/P is amenable, these spectra coincide as sets.

The examples that admit explicit calculations of the spectra have yielded surprising results. It turned out that the spectra of automata may be the unions of a finite number of intervals as well as totally disconnected sets, for example a Cantor set [7]. On the other hand, in [30], using a two-state automaton, we calculated the spectral measure of the Laplace operator on the Cayley graph of the lamplighter group. Moreover, it turned out that the spectral measure of this operator is discrete; this result is absolutely unexpected for the theory of random walks on groups. A direct consequence of the result of [30] is the negative solution to the conjecture of Atiyah on the denominators of rational values of L^2-cohomological invariants of manifolds [25].

The study of the groups of finite automata, dynamical systems, and the graphs and spectra generated by finite automata has passed the initial stage of its development. Many problems still remain unsolved, and new fields of applications have opened. We hope that this paper will draw attention of the reader to the problems considered.

2. ASYNCHRONOUS AUTOMATA

2.1. Word spaces. Let X be a finite set, $|X| > 1$; we call this set an *alphabet*.

For a given alphabet X, we denote by X^* a free monoid generated by the set X. The elements of the monoid X^* are represented as words $x_1x_2\dots x_n$ (including the empty word $\varnothing$). If $w = x_1x_2\dots x_n \in X^*$, then $|w| = n$ is the length of the word w. The length of the empty word is equal to zero.

Along with finite words from X^*, we also consider infinite sequences (infinite words) of the form $x_1x_2x_3\dots$, where $x_i \in X$. The set of such infinite words is denoted by X^ω.

For arbitrary $w \in X^*$ and $u \in X^* \cup X^\omega$, we naturally define the concatenation (product) $wu \in X^\omega$.

A word $w \in X^*$ is the *beginning*, or *prefix*, of a word $u \in X^*$ ($\in X^\omega$) if $u = wv$ for a certain $v \in X^*$ ($\in X^\omega$); in this case, we denote $v = u - w$. If u is the beginning of the word w, then we set $u - w = \varnothing$; otherwise, $u - w$ is indeterminate.

For an arbitrary set of words $A \subseteq X^* \cup X^\omega$, there is a uniquely defined longest common prefix (the longest beginning) of words from A, which we denote by $\mathcal{P}(A)$. Note that $\mathcal{P}(A)$ is infinite if and only if the set A consists of a single infinite word.

The set X^ω is an infinite Cartesian product $X^{\mathbb{N}}$; one can introduce on this product the topology of the direct Tikhonov product of finite discrete topological spaces X (the topology of pointwise convergence). In this topology, X^ω is homeomorphic to the Cantor set. Thus, the topological type of this space is independent of X.

For any finite word $w \in X^*$, the set $wX^\omega \subseteq X^\omega$ of all words beginning with w is clopen (open and closed) in the given topology, while the family of all such sets $\{wX^\omega\colon\ w \in X^*\}$ is the base of the topology.

Note that the sets w_1X^ω and w_2X^ω have a nonempty intersection if and only if one of the words w_1 or w_2 is the beginning of the other; then, the set corresponding to the longer of the words is a subset of the other set.

Let $\widetilde{U}_w$ be a set of all finite and infinite words with the prefix $w \in X^*$. The family of all sets $\{\widetilde{U}_w\colon\ w \in X^*\}$ is also the base of open sets of compact topology on $X^* \cup X^\omega$. In this topology, the set X^* is a countable set of isolated points whose closure is given by the set $X^* \cup X^\omega$, while the induced topology on X^ω coincides with that introduced above. An infinite word $u \in X^\omega$ belongs to the closure of the set $A \subseteq X^*$ if and only if an infinite number of prefixes of the word u are the prefixes of certain words from A.

A word $v \in X^\omega$ is called *almost periodic* if it has the form $uwww\dots$, where u and w are finite words.

For any decreasing sequence of positive numbers $\overline{\lambda} = (\lambda_n)_{n=0}^\infty$ such that $\lim_{n\to\infty} \lambda_n = 0$, there exists a metric

$$d_{\overline{\lambda}}(w_1, w_2) = \lambda_n$$

defined on the space X^ω, where $n = |\mathcal{P}(w_1, w_2)|$ is the length of the longest common prefix of the words w_1 and w_2 (the distance between identical words is equal to zero).

This metric is an ultrametric; i.e., the following inequality holds for any $w_1, w_2, w_3 \in X^\omega$:

$$d_{\overline{\lambda}}(w_1, w_3) \le \max(d_{\overline{\lambda}}(w_1, w_2), d_{\overline{\lambda}}(w_2, w_3)).$$

In this case, the set wX^ω of infinite words beginning with w is a ball of radius $\lambda_{|w|}$ with the center at an arbitrary point $u \in wX^\omega$.

2.2. Transducers.

Definition 2.1. An *asynchronous automaton* (a *generalized sequential machine*, according to [18]) is a set $A = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q, \pi, \lambda \rangle$, where

(1) X_{I} and X_{O} are finite sets (respectively, the *input and output alphabets*),

(2) Q is a set (the *set of internal states of the automaton*),

(3) $\pi\colon X_{\mathrm{I}} \times Q \to Q$ is a mapping (*transition function*), and

(4) $\lambda\colon X_{\mathrm{I}} \times Q \to X_{\mathrm{O}}^*$ is a mapping (*output function*).

The cardinality of the set of states of an automaton is called the cardinality of the automaton. In particular, automaton A is *finite* if $|Q| < \infty$.

If every value of the function $\lambda(\,\cdot\,,\,\cdot\,)$ is a one-letter word, then automaton A is called a *synchronous* automaton.

The functions λ and π can be continued to the set $X_{\mathrm{I}}^* \times Q$ according to the following recurrent rules:

$$\pi(\varnothing, q) = q, \qquad \pi(xw, q) = \pi(w, \pi(x, q)), \tag{1}$$

$$\lambda(\varnothing, q) = \varnothing, \qquad \lambda(xw, q) = \lambda(x, q)\lambda(w, \pi(x, q)), \tag{2}$$

where $x \in X_{\mathrm{I}}$, $q \in Q$, and $w \in X_{\mathrm{I}}^*$ are arbitrary elements.

The above formulas are also equivalent to the formulas

$$\begin{aligned} \pi(\varnothing, q) &= q, \qquad \pi(wx, q) = \pi(x, \pi(w, q)), \\ \lambda(\varnothing, q) &= \varnothing, \qquad \lambda(wx, q) = \lambda(w, q)\lambda(x, \pi(w, q)). \end{aligned} \tag{3}$$

The automaton A_{q_0} with a fixed *initial* state $q_0 \in Q$ is called an *initial automaton.* Any initial automaton defines a function $\lambda(\,\cdot\,, q_0)\colon X_{\mathrm{I}}^* \to X_{\mathrm{O}}^*$ that specifies the *action* of the automaton A_{q_0} on finite words, which is also denoted as $\lambda(w, q_0) = w^{A_{q_0}}$.

For an initial automaton A_{q_0} and a word $v \in X_{\mathrm{I}}^*$, the state $\pi(v, q_0)$ is called the *state of the automaton in the word v.*

Finite automata can be represented as labeled directed graphs (the Moore diagrams). The vertices of such a graph correspond to the states of the automaton, and, for every symbol $x \in X_{\mathrm{I}}$ of the input alphabet, an arrow labeled by $x|\lambda(x, q)$ starts from state $q \in Q$ to $\pi(x, q)$. If the automaton is initial, we draw the initial state in the form of a double circle on its Moore diagram or denote this state by the same letter as the automaton. To find out the action of the automaton on the word w, we should move, starting from the labeled vertex, along the arrows of the graph so that the word w reads on the left parts of the labels along the arrows; then, the product of all right parts of the labels will be equal to $\lambda(w, q_0)$.

The state $q \in Q$ of the initial automaton A_{q_0} is called *accessible* if there exists a word $w \in X_{\mathrm{I}}^*$ such that $\pi(w, q_0) = q$. An automaton is called *accessible* if all its states are accessible. Let Q' be a set of all accessible states of an asynchronous automaton A. Then, the automaton $A'_{q_0} = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q', \pi', \lambda' \rangle_{q_0}$, where the mappings π' and λ' are the restrictions of the corresponding mappings for the automaton A onto $X_{\mathrm{I}} \times Q'$, acts on words in the same way as the automaton A_{q_0}.

An initial asynchronous automaton A_{q_0} is called *nondegenerate* (*almost positive*, according to [18]) if, for any infinite word $w \in X_{\mathrm{I}}^{\omega}$, the recurrent formulas (2) uniquely define an infinite word $\lambda(w, q_0)$. The word $\lambda(w, q_0)$ is called the *image* of the word w under the action of the automaton and is denoted by $w^{A_{q_0}}$ as in the case of finite words.

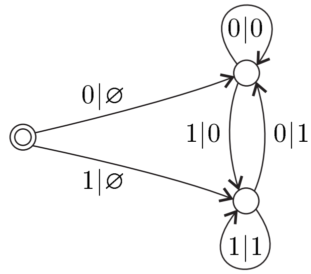


Fig. 1. Example of an asynchronous automaton

An initial automaton A_{q_0} is nondegenerate if and only if there do not exist any accessible state $q \in Q$ and an infinite word $w \in X_{\mathrm{I}}^{\omega}$ such that, for an arbitrary prefix u of the word w, the word $\lambda(u, q)$ is empty. In particular, an arbitrary synchronous automaton is nondegenerate.

The following nondegeneracy criterion is valid.

Proposition 2.1. *An initial automaton is nondegenerate if and only if, for any accessible state $q \in Q$ of this automaton, there exists only a finite set of words $w \in X_{\mathrm{I}}^{*}$ such that the word $\lambda(u, q)$ is empty.*

A mapping $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ is said to be defined by a (nondegenerate) automaton A_{q_0} if $f(w) = \lambda(w, q_0)$ for any $w \in X_{\mathrm{I}}^{\omega}$. The mapping defined by a nondegenerate automaton is called the action (on infinite words) of this automaton.

Note that different asynchronous automata may define equal mappings on the set of infinite words. For example, the automaton over the two-letter alphabet $\{0, 1\}$ that is depicted in Fig. 1 removes the final letter from every finite word; therefore, its action on infinite words is trivial, i.e., coincides with the action of the automaton that acts trivially on finite words.

In this relation we give the following definition.

Definition 2.2. Two initial asynchronous automata are called *equivalent* if they define equal mappings on the space of finite words X_{I}^{*}.

Two initial asynchronous automata are called *ω-equivalent* if they define equal mappings on the space of infinite words X_{I}^{ω}.

The proof of the following assertion is straightforward.

Proposition 2.2. *The image of an almost periodic word under the action of a finite asynchronous automaton is an almost periodic word.*

As in the classical theory of automata, there exists a canonical automaton in the class of ω-equivalent automata. To describe this automaton more conveniently, we define the concept of a *modified* automaton by changing the definition of the output function as follows:

(4) an output function is the mapping $\lambda\colon X_{\mathrm{I}} \times Q \to X_{\mathrm{O}}^{*} \cup X_{\mathrm{O}}^{\omega}$; here, if $\lambda(x, q) \in X_{\mathrm{O}}^{\omega}$, the word $\lambda(w, \pi(x, q))$ is empty for all words $w \in X_{\mathrm{I}}^{*}$.

In this case, we assume that a modified automaton is finite if it has a finite number of states and each word $\lambda(x, q)$ is either finite or almost periodic.

Thus, we allowed an asynchronous automaton to output an infinite word in a single step; however, if the automaton outputs such a word, then, after that, it will output only empty words. In this case, the recurrent formulas (2) also define the action of a nondegenerate modified automaton on infinite words.

Proposition 2.3. *The class of mappings defined by (finite) modified automata coincides with the class of mappings defined by ordinary (finite) asynchronous automata.*

Proof. Indeed, let f be defined by a modified asynchronous automaton. For every pair $x \in X$, $q \in Q$ such that $w = \lambda(x,q) \in X_{\mathrm{O}}^{\omega}$, we add an infinite sequence of new states $q_1, q_2, \dots$, define $\pi(x,q) = q_1$ and $\pi(y,q_i) = q_{i+1}$ for any $i \geq 1$ and $y \in X_{\mathrm{I}}$, set $\lambda(y,q_i)$ equal to the ith letter of the word w for any $y \in X_{\mathrm{I}}$, and redefine $\lambda(x,q) = \varnothing$. It can be easily shown that, as a result, we obtain an asynchronous automaton that defines the same mapping.

In the case of a finite modified automaton, for every pair $x \in X$, $q \in Q$ such that $w = \lambda(x,q) \in X_{\mathrm{O}}^{\omega}$, we add a new state p, define $\pi(x,q) = p$ and $\pi(y,p) = p$ for any $y \in X_{\mathrm{I}}$, and set $\lambda(x,q) = v$ and $\lambda(y,p) = u$ for any $y \in X_{\mathrm{I}}$, where $w = vuuu\dots$. Note also that, if, for a finite automaton, the word $\lambda(uw,q)$ is independent of $w \in X_{\mathrm{I}}^{\omega}$, then it is almost periodic according to Proposition 2.2. □

Henceforth, we will deal only with the definition of modified automaton.

The theory of asynchronous automata is considered in a part of the monograph [18], where the author mainly deals with the action of these automata on finite words. In addition to the theory of groups, asynchronous automata have found application in the theory of coding [78, 68]; they also allow one to perform various arithmetic calculations [18, 38].

2.3. Semigroups of asynchronous automata. Let $A_1 = \langle X_1, X_2, Q_1, \pi_1, \lambda_1 \rangle$ and $A_2 = \langle X_2, X_3, Q_2, \pi_2, \lambda_2 \rangle$ be two automata.

Let us construct a new automaton $B = A_1 * A_2$ with the set of states $Q_1 \times Q_2$ and the transition π and output λ functions defined, respectively, by the equalities

$$\pi(x,(s_1,s_2)) = (\pi_1(x,s_1), \pi_2(\lambda_1(x,s_1),s_2)),$$
$$\lambda(x,(s_1,s_2)) = \lambda_2(\lambda_1(x,s_1),s_2).$$

Note that, in general, $\lambda_1(x,s_1)$ is a word rather than a letter. The meaning of these equalities is that we connect the output of the first automaton to the input of the second and process information successively. The automaton $B = A_1 * A_2$ is called a *composition* of the automata A_1 and A_2. The automaton $A_1 * A_2$ with the initial state (q_1,q_2) is called the composition of initial automata A_{1,q_1} and A_{2,q_2}.

It is clear that, if w is a finite or infinite word, then $w^{(A_1 * A_2)_{(q_1,q_2)}} = \left(w^{A_{1,q_1}}\right)^{A_{2,q_2}}$.

Thus, the set of all mappings $f\colon X^{\omega} \to X^{\omega}$ defined by initial asynchronous automata forms a semigroup with respect to the composition. This semigroup is called a *semigroup of asynchronous automata* and is denoted by $\mathcal{C}(X^{\omega})$.

Theorem 2.4. *The mapping $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ is continuous if and only if it is defined by a certain nondegenerate asynchronous automaton.*

Proof. Let A_{q_0} be a nondegenerate asynchronous automaton. For any $w \in X_{\mathrm{I}}^{*}$, $\lambda(w,q_0)$ is the beginning of the word $\lambda(wu,q_0)$ for any $u \in X_{\mathrm{I}}^{\omega}$. This entails the continuity of the mapping $\lambda(\,\cdot\,,q_0)$.

Let $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ be a continuous mapping. Let us construct an automaton $A = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q, \pi, \lambda \rangle$ that defines this mapping. Set $Q = X_{\mathrm{I}}^{*}$, $q_0 = \varnothing$, and, for any $x \in X_{\mathrm{I}}^{*}$ and $w \in Q$, let $\pi(x,w) = wx$. For any $w \in X_{\mathrm{I}}^{*}$, denote

$$l(w) = \mathcal{P}\{f(wu)\colon\ u \in X_{\mathrm{I}}^{\omega}\}.$$

Since the mapping f is continuous, for any $x_1x_2\ldots \in X_{\mathrm{I}}^{\omega}$, the length of the word $w_n = l(x_1x_2\ldots x_n)$ does not decrease and tends to infinity as $n \to \infty$, and the words w_n are the beginnings of the word $f(x_1x_2\ldots)$.

Let $\lambda(x,w) = l(wx) - l(w)$. Then, the automaton constructed defines f.

Note that, for any state q of the constructed automaton, the longest common prefix $\mathcal{P}\{\lambda(xw,q)\colon w \in X_{\mathrm{I}}^{\omega}\}$ is equal to $\lambda(x,q)$. $\square$

In the general case, an asynchronous automaton defining a continuous mapping is infinite.

Thus, the semigroup $\mathcal{C}(X^{\omega})$ is isomorphic to the semigroup of all continuous transformations of the Cantor set X^{ω}. Hence, the isomorphic type of this semigroup is independent of the cardinality of the alphabet.

If the mapping $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ defined by a certain asynchronous automaton is bijective, then this mapping is a homeomorphism, and the inverse mapping f^{-1} is also defined by a certain asynchronous automaton. Therefore, the set of all bijective mappings defined by asynchronous automata with the input and output alphabet X forms a group. This group is called a *group of asynchronous automata* and is denoted by $\mathcal{H}(X^{\omega})$. This group is also independent of X since it is isomorphic to the group of homeomorphisms of the Cantor set.

2.4. Finite automata. Continuous mappings defined by finite initial asynchronous automata have the following characterization.

Definition 2.3. Let $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ be a continuous mapping and $w \in X_{\mathrm{I}}^{*}$ be a finite word. The *restriction* of the mapping f in the word w is the mapping $f|_w\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ defined by the equality

$$f(wu) = vf|_w(u),$$

where $v = \mathcal{P}\{f(wu)\colon u \in X_{\mathrm{I}}^{\omega}\}$.

If the word $f(wu)$, $u \in X_{\mathrm{I}}^{\omega}$, is independent of u, the restriction $f|_w$ is said to be *empty* and is denoted as $f|_w(u) = \varnothing$.

Theorem 2.5. *A continuous mapping $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ is defined by a certain finite nondegenerate asynchronous automaton if and only if it has a finite number of different restrictions and, for any word $w \in X_{\mathrm{I}}^{*}$ such that the restriction $f|_w$ is empty, the word $f(wu)$ is almost periodic for any $u \in X_{\mathrm{I}}^{\omega}$.*

Proof. Let A be a finite nondegenerate asynchronous automaton. For every $w \in X_{\mathrm{I}}^{*}$, the restriction $f|_w$ is uniquely defined by the state $\pi(w,q_0)$ since

$$\lambda(wu,q_0) = \lambda(w,q_0)\lambda(u,\pi(w,q_0)) = \lambda(w,q_0)vf|_w(u),$$

where $v = \mathcal{P}\{\lambda(u,\pi(w,q_0))\colon u \in X_{\mathrm{I}}^{\omega}\}$. Hence, the set of different restrictions of a mapping defined by a finite automaton is finite.

Conversely, let f be a mapping with a finite set of restrictions. Denote this set by Q. Construct an automaton with the set of internal states Q and the initial state $f|_{\varnothing} = f$ so that its action on the space X_{I}^{ω} coincides with the action of the mapping f.

Put $\pi(x, f|_w) = f|_{wx}$. The mapping π is well defined since $f|_{wx} = (f|_w)|_x$.

Define the output function as

$$\lambda(x,f|_w) = \mathcal{P}\{f|_w(xu)\colon u \in X_{\mathrm{I}}^{\omega}\} = \mathcal{P}\{f(wxu)\colon u \in X_{\mathrm{I}}^{\omega}\} - \mathcal{P}\{f(wu)\colon u \in X_{\mathrm{I}}^{\omega}\}.$$

(Note that, if the restriction $f|_{wx}$ is empty, then $\lambda(x,f|_w)$ is infinite.) The automaton obtained defines the mapping f. $\square$

Definition 2.4. A mapping $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ is called *rational* if it is defined by a certain finite asynchronous automaton.

Since the set of states of the composition $A_1 * A_2$ of automata is equal to the Cartesian product of the sets of states of the automata A_1 and A_2, the composition of finite automata also is a finite automaton. Therefore, the set of all finite automata forms a semigroup with respect to the composition. This semigroup is called a *semigroup of finite asynchronous automata* and is denoted by $\mathcal{F}(X^{\omega})$.

2.5. Reduced automata. One of the basic facts in the theory of finite automata is the existence of a unique automaton in the class of equivalent automata that has the minimal number of states and is called the *minimal* automaton. The relevant algorithm is described in [18].

Here, we consider the problem of the existence of a canonical asynchronous automaton that is ω-equivalent to a given automaton. The determination of such an automaton is called a *reduction.* This procedure will be described in Subsection 2.8.1.

Definition 2.5. A state $q \in Q$ of an asynchronous automaton A is called a *state with incomplete response* if, for a certain $x \in X$, the longest common prefix of the words $\lambda(xw, q)$, $w \in X_{\mathrm{I}}^{\omega}$, is different from $\lambda(x, q)$.

Note that the automaton constructed during the proof of Theorem 2.4 does not contain any states with incomplete response. It can be readily shown that the automaton constructed during the proof of Theorem 2.5 also does not contain such states. Hence, the following assertion holds.

Proposition 2.6. *For any (finite) asynchronous automaton, there exists a (finite) ω-equivalent asynchronous automaton that has no states with incomplete response.*

To asynchronous automata without states with incomplete response, one can apply the results of the classical theory of minimization of generalized sequential machines presented in [18] since the following assertion holds.

Proposition 2.7. *Two initial automata without states with incomplete response are ω-equivalent if and only if they are equivalent.*

Proof. It suffices to prove that the action of an initial automaton A_{q_0} without states with incomplete response on the set X_{I}^{*} of finite words is uniquely defined by its action on X_{I}^{ω}. However, it follows from Definition 2.5 that, if an automaton does not have any states with incomplete response, then, for an arbitrary finite word $v \in X_{\mathrm{I}}^{*}$, we have $v^{A_{q_0}} = \mathcal{P}\{(vw)^{A_{q_0}} : w \in X_{\mathrm{I}}^{\omega}\}$. $\square$

On the last step of reduction, equivalent states are identified, where two states $q_1, q_2 \in Q$ of an automaton A are called *equivalent* if, for any word $w \in X_{\mathrm{I}}^{*}$, the words $\lambda(w, q_1)$ and $\lambda(w, q_2)$ coincide.

Definition 2.6. An asynchronous automaton A is called *reduced* if it does not have any states with incomplete response and equivalent states.

An asynchronous initial automaton A_{q_0} is called *reduced* if it is accessible and the corresponding noninitial automaton is reduced.

Two automata $A' = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q', \pi', \lambda' \rangle$ and $A'' = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q'', \pi'', \lambda'' \rangle$ are called *isomorphic* if there exists a bijection $\phi\colon Q_1 \to Q_2$ that commutes with the transition and output functions, i.e., such that

$$\pi''(x, \phi(q)) = \phi(\pi'(x, q)),$$
$$\lambda''(x, \phi(q)) = \lambda'(x, q).$$

If the automata are initial, then the bijection must carry over the initial state of the first automaton to the initial state of the second.

The following proposition shows that a reduced automaton ω-equivalent to a given automaton is defined uniquely.

Proposition 2.8. *If reduced initial automata are ω-equivalent, then they are isomorphic.*

Proof. If an automaton is reduced, its states are in one-to-one correspondence with the restrictions of the mapping defined by this automaton. In this case, the output and transition functions are uniquely defined by the restrictions. Therefore, a reduced automaton is defined uniquely up to an isomorphism by the mapping that it defines. □

Thus, the reduction algorithm will allow us to find out whether or not given automata are ω-equivalent.

2.6. Rational sets.

Definition 2.7. A subset $L \subseteq X^\omega$ is called *rational* if there exists a finite directed graph with the labeled initial vertex and the arrows labeled by the elements of the alphabet X, such that the labels of the arrows starting from the same vertex are pairwise different and the set of all words that can be obtained by successive concatenation of labels on the paths of the graph with the origin at the labeled vertex coincides with L.

The graph involved in this definition is called a (*deterministic*) *automaton that recognizes the set* L.

Obvious examples of rational sets are given by the sets X^ω themselves, by finite sets of almost periodic words, and by the spaces of unilateral shifts of finite type.

If we set X^* instead of X^ω in Definition 2.7, then this definition is one of possible definitions of a rational, or, in other, more frequently used terms, a regular prefix-closed set (or language).

Recall that a regular language $L \subseteq X^*$ is called *prefix-closed* if, for every $w \in L$, every prefix of the word w belongs to L (see [18, 17]).

It is obvious that every rational set is closed and, hence, compact. Moreover, one can derive the following characterization of rational sets directly from the definitions.

Proposition 2.9. *A set $L \subseteq X^\omega$ is rational if and only if it is a closure of a certain prefix-closed regular language $L_0 \subseteq X^*$.*

Along with the deterministic automata that recognize the sets, one can also consider *nondeterministic automata*, i.e., finite directed graphs in which the arrows are labeled by words rather than by letters (arrows with identical labels may start from the same vertex). The set of infinite words that are read on infinite paths with a given origin in a nondeterministic graph is also a rational set by virtue of a similar result for regular languages (see [18]) and Proposition 2.9. Moreover, there exists an algorithm for determining the minimal (i.e., with the least possible number of vertices) deterministic automaton that recognizes a set defined by a finite nondeterministic automaton.

Note that two minimal deterministic automata that define the same set are isomorphic; therefore, this algorithm allows one to find out whether or not two nondeterministic automata define identical sets.

Thus, Proposition 2.9 implies that there exists an algorithm that determines, by two given automata, whether or not the rational sets recognized by these automata coincide.

In addition, the following proposition holds.

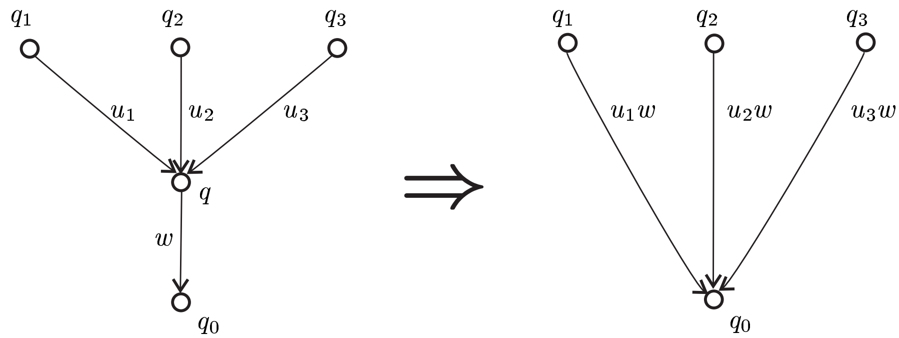


Fig. 2. Reconstruction of an automaton at the first stage

Proposition 2.10. *The intersection of a finite number of rational sets is a rational set. There exists an algorithm that constructs, by two automata that recognize these rational sets, an automaton that recognize the intersection of these sets.*

The relation between finite asynchronous automata and rational sets is illustrated by the following proposition.

Proposition 2.11. *A set $L \subseteq X^\omega$ is rational if and only if it is the image of the set X_{I}^ω under the action of a certain rational mapping $f\colon X_{\mathrm{I}}^\omega \to X^\omega$; here, one can take any alphabet of cardinality ≥ 2 as X_{I}. If L has no isolated points, then the mapping f can be assumed to be an injection.*

Proof. Since the set X_{I}^ω is rational for any finite alphabet X_{I}, it suffices to prove the proposition for the case $|X_{\mathrm{I}}| = 2$, whence one can easily derive the general case.

Suppose that we have the Moore diagram of an asynchronous initial automaton. In order to obtain, from this automaton, a (nondeterministic) automaton that recognizes the image of the corresponding mapping, one should replace each label $x|w$ of the graph of the asynchronous automaton by the label w, thus obtaining the graph of a new automaton.

Conversely, let a closed set $L \subseteq X^\omega$ be rational and let Γ be a graph that is a deterministic automaton that recognizes the set L. Assume that L has no isolated points. Then, there does not exist a vertex of the graph Γ from which only a single infinite path starts. If q is a vertex of the graph Γ from which only one arrow emanates, then, making a change in Γ as is shown in Fig. 2, we reduce the number of vertices with a single emanating arrow. Note that $q \neq q_0$; otherwise, only one infinite path will start from q.

The infinite paths in the new graph are obtained from the paths of the initial graph by deleting all entries of the vertex q. This correspondence between the infinite paths of the new and original graphs is one-to-one and preserves infinite words read on the paths. Therefore, if different words are read on any two infinite paths with a common origin in the first graph, the same will be valid for the second graph.

Proceeding in this way, we obtain the automaton, that recognizes the set L, in which more than one arrow emanates from each vertex. In this case, even though the automaton may not be deterministic any longer, different words will be read on different paths with a common origin.

When L has isolated points, we leave the recognizing automaton unchanged.

Let us transform the graph that recognizes the set L into the Moore diagram of the automaton that defines the mapping f. For this purpose, we replace each vertex of the graph with several arrows emanating from it by a graph of an asynchronous automaton, as is shown in Fig. 3.

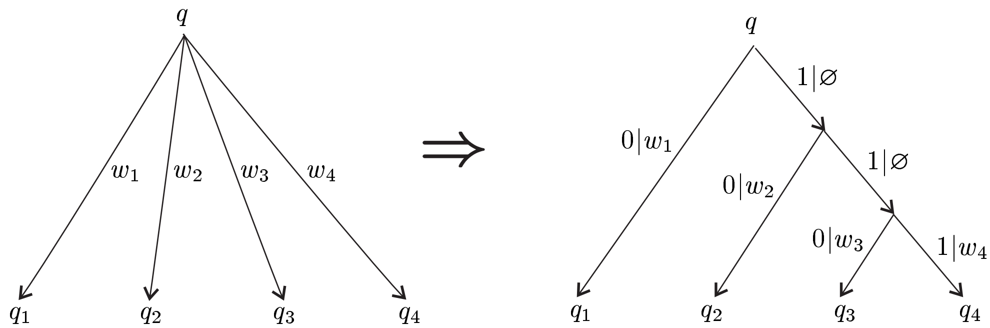


Fig. 3. Reconstruction of an automaton at the second stage

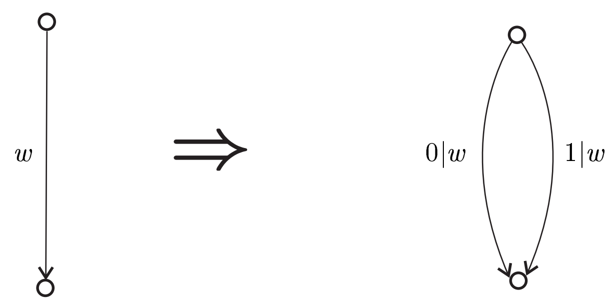
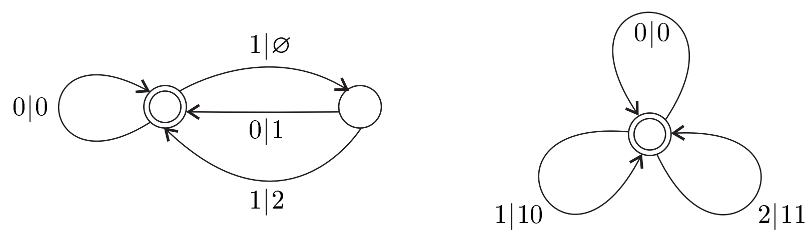


Fig. 4. Reconstruction of an automaton at the third stage



(a) (b)

Fig. 5. The automata implementing a bijection between the spaces of two-letter and three-letter sequences

Note that some of the vertices $q, q_1, q_2, \ldots$ in Fig. 3 on the left may coincide (then they must also coincide after the replacement).

When L has isolated points, there are vertices in the graph from which only one arrow emanates. In this case, the reconstruction occurs as is shown in Fig. 4.

If the set L has no isolated points, we transformed the recognizing automaton into the graph in which different words are read on different paths originating from the initial vertex. Therefore, under the described transformation of this automaton into the Moore diagram of the asynchronous automaton, the corresponding rational mapping will be injective. □

Corollary 2.12. *For any two finite alphabets X_{I} and X_{O}, there exists a finite automaton $R(X_{\mathrm{I}}, X_{\mathrm{O}})$ that defines a continuous bijective mapping $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$.*

These automata can be constructed explicitly. For example, the automaton $R(X_{\mathrm{I}}, X_{\mathrm{O}})$ that implements the bijection $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ for $X_{\mathrm{I}} = \{0,1\}$ and $X_{\mathrm{O}} = \{0,1,2\}$ is shown in Fig. 5a. Figure 5b represents the inverse automaton $R(X_{\mathrm{O}}, X_{\mathrm{I}})$.

2.7. Group of rational homeomorphisms. If $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ is a continuous bijection, then, since X_{I}^{ω} and X_{O}^{ω} are compact spaces, the inverse mapping f^{-1} is also continuous. Therefore, f^{-1} is also defined by an asynchronous automaton.

Using Theorem 2.5, we can prove the following proposition.

Proposition 2.13. *If $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ is a bijective mapping defined by a finite automaton, then the inverse mapping f^{-1} is also defined by a finite automaton.*

Thus, for any finite alphabet X, the set $\mathcal{Q}(X^{\omega})$ of all rational homeomorphisms $f\colon X^{\omega} \to X^{\omega}$ is a group.

Proposition 2.12 implies the following assertion.

Proposition 2.14. *The isomorphic types of the group $\mathcal{Q}(X^{\omega})$ and the semigroup $\mathcal{F}(X^{\omega})$ are independent of the alphabet X, $|X| > 1$. Moreover, the groups $\mathcal{Q}(X_{\mathrm{I}}^{\omega})$ and $\mathcal{Q}(X_{\mathrm{O}}^{\omega})$ and the semigroups $\mathcal{F}(X_{\mathrm{I}}^{\omega})$ and $\mathcal{F}(X_{\mathrm{O}}^{\omega})$ are conjugate by the automaton $R(X_{\mathrm{I}}, X_{\mathrm{O}})$.*

This proposition shows that a *semigroup of rational continuous mappings* and a *group of rational homeomorphisms* of the Cantor set, which are denoted by $\mathcal{F}$ and $\mathcal{Q}$, respectively, are naturally defined.

Identifying ω-equivalent automata, we will sometimes refer to the elements of the semigroup $\mathcal{F}$ and group $\mathcal{Q}$ as automata.

2.8. Algorithms for calculations with finite automata.

2.8.1. *Reduction.* For efficient calculations with finite asynchronous automata, one should be able, first of all, to determine whether two automata define equal rational mappings.

Theorem 2.15. *There exists an algorithm that determines whether or not two given finite asynchronous automata are ω-equivalent.*

Proof. By Proposition 2.8, it suffices to be able to determine a reduced automaton that is ω-equivalent to a given one.

Let $A = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q, \lambda, \pi\rangle$ be a certain finite automaton.

For $v \in X_{\mathrm{I}}^{*}$ and $q \in Q$, denote

$$\Lambda(v,q) = \mathcal{P}\{\lambda(vw,q)\colon\ w \in X_{\mathrm{I}}^{\omega}\}.$$

The automaton A is an automaton without states with incomplete response if and only if $\Lambda(v,q) = \lambda(v,q)$. In the general case, $\lambda(v,q)$ is only the beginning of the word $\Lambda(v,q)$.

The first stage of the reduction is the construction of an automaton without states with incomplete response that is ω-equivalent to A_q. First, for every state $q \in Q$, we determine $\Lambda(\varnothing, q)$. For this purpose, we calculate the image of the set X_{I}^{ω} under the action of the automaton A_q as a rational set and then determine the longest common prefix of all its elements using the classical methods of the theory of regular languages. After that, we can apply the following proposition.

Proposition 2.16. *Let $A_{q_0} = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q, \pi, \lambda\rangle_{q_0}$ be a finite initial automaton. Let a set Q' be given by $Q' = Q \cup \{q_{-1}\}$, where q_{-1} is a new state. Set*

$$\pi(x, q_{-1}) = \pi(x, q_0),$$
$$\lambda(x, q_{-1}) = \Lambda(x, q_0)$$

for all $x \in X_{\mathrm{I}}$. Redefine the output function by the equality

$$\lambda'(x,q) = \Lambda(x,q) - \Lambda(\varnothing, q)$$

for all $q \in Q$ and $x \in X_{\mathrm{I}}$.

Then, the automaton $A' = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q', \pi, \lambda' \rangle$ with the initial state q_{-1} will be an automaton without states with incomplete response, which is ω-equivalent to the automaton A_{q_0}.

In addition, for every $w \in X_{\mathrm{I}}^{\omega}$, the following equality holds:

$$\lambda'(w, q_{-1}) = \Lambda(w, q_0).$$

Proof. It suffices to prove the last equality. Let us apply the induction on the length of the word w.

For the words of length 0 and 1, the equality is fulfilled. Suppose that it is valid for all words of length n. Let wx be a word of length $n+1$ and $x \in X_{\mathrm{I}}$.

Note that, for an arbitrary nonempty word v, the state $\pi(v, q_{-1})$ coincides with the state $\pi(v, q_0)$. From the induction hypothesis we obtain

$$\lambda'(wx, q_{-1}) = \lambda'(w, q_{-1})\lambda'(x, \pi(w, q_0)) = \Lambda(w, q_0)[\Lambda(x, \pi(w, q_0)) - \Lambda(\varnothing, \pi(w, q_0))] = \Lambda(wx, q_0)$$

since

$$\Lambda(wx, q_0) = \lambda(w, q_0)\Lambda(x, \pi(w, q_0)),$$
$$\Lambda(w, q_0) = \lambda(w, q_0)\Lambda(\varnothing, \pi(w, q_0)). \quad \square$$

The next step of reduction is the elimination of all inaccessible states from the Moore diagram of the new automaton together with all incoming and outgoing arrows. It is clear that all inaccessible states can be found in a finite number of steps.

By Proposition 2.7, the automaton obtained can be minimized (i.e., one can identify the equivalent states) using the minimization algorithm for generalized sequential machines that was described in [18]. Theorem 2.15 is proved. $\square$

As a corollary to Proposition 2.16, we obtain the following property of actions of ω-equivalent automata on finite words.

Proposition 2.17. *If A_1 and A_2 are ω-equivalent initial finite asynchronous automata, then there exists a positive number N such that the following inequality holds for any $w \in X_{\mathrm{I}}^*$:*

$$\Big| |w^{A_1}| - |w^{A_2}| \Big| \le N.$$

Proof. It suffices to prove the assertion in the case when one of the automata is reduced. Under the hypotheses of Proposition 2.16, the set of all possible differences $\Lambda(w, q_0) - \lambda(w, q_0) = w^{A'_{q_{-1}}} - w^{A_{q_0}}$ is finite since $\Lambda(w, q_0) - \lambda(w, q_0) = \Lambda(\varnothing, \pi(w, q_0))$. Hence, there exists N such that $\big| |w^{A_{q_0}}| - |w^{A'_{q_{-1}}}| \big| \le N$. $\square$

2.8.2. *Determination of the inverse.* The following assertion from [18] shows that the mapping $f\colon X^* \to X^*$ defined by an automaton is invertible only in the case of a synchronous automaton.

Proposition 2.18. *If the mapping $f\colon X_{\mathrm{I}}^* \to X_{\mathrm{O}}^*$ defined by an automaton A_{q_0} is invertible and $|X_{\mathrm{I}}| \le |X_{\mathrm{O}}|$, then it is defined by a synchronous automaton.*

If $|X_{\mathrm{I}}| > |X_{\mathrm{O}}|$, then there exists a bijective mapping $f\colon X_{\mathrm{I}}^* \to X_{\mathrm{O}}^*$ defined by an asynchronous automaton (see example 2.3 in [18]); however, by Proposition 2.18, the inverse mapping cannot be determined by an automaton.

Thus, from the viewpoint of applications in the theory of groups and theory of dynamical systems, in the case of asynchronous automata, it seems more natural to investigate the action of automata on infinite words. The case of mappings defined by synchronous automata will be considered separately in Section 3.

The problems of invertibility of mappings defined by asynchronous automata were investigated in [77, 78].

Let us construct an algorithm that allows one to determine whether a given finite automaton defines an invertible mapping.

In order to determine whether the mapping $f\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ defined by a finite automaton is a surjection, one has to determine the image $f(X_{\mathrm{I}}^{\omega})$, which is a rational set; i.e., one should find a deterministic automaton that recognizes this image, and then find out whether this set coincides with the set X_{O}^{ω}. It is well known that there exists an algorithm for rational sets that solves this problem (see [18]).

Note that, for an accessible automaton A_{q_0}, the injectiveness of the mapping defined by this automaton implies the injectiveness of all mappings defined by the automata A_q for all states q of the automaton A.

Moreover, the following lemma holds, by which one can efficiently determine whether the mapping defined by a finite automaton is injective.

Lemma 2.19. *A mapping defined by an accessible automaton A_{q_0} is injective if and only if the sets*

$$\{U_{x,q} = \{\lambda(xw, q)\colon\ w \in X_{\mathrm{I}}^{\omega}\}\}_{x \in X_{\mathrm{I}}}$$

are pairwise disjoint for every state $q \in Q$.

Proof. Assume that there exist two different infinite words $w_1 = x_1x_2\dots, w_2 = y_1y_2\dots \in X_{\mathrm{I}}^{\omega}$ such that $f(w_1) = f(w_2)$. Let $x_i = y_i$ for any $1 \le i < n$ and $x_n \ne y_n$. Consider the state $q = \pi(x_1x_2\dots x_{n-1}, q_0)$. Then,

$$\begin{aligned} f(w_1) &= \lambda(x_1x_2\dots x_{n-1}, q_0)\lambda(x_nx_{n+1}\dots, q) \\ = f(w_2) &= \lambda(x_1x_2\dots x_{n-1}, q_0)\lambda(y_ny_{n+1}\dots, q); \end{aligned}$$

hence, the sets $U_{x_n,q}$ and $U_{y_n,q}$ intersect.

Conversely, if $U_{x,q} \cap U_{y,q} \ne \varnothing$ for a certain accessible state $q = \pi(v, q_0)$ and different letters $x, y \in X_{\mathrm{I}}$, then there exist infinite words $u_1, u_2 \in X_{\mathrm{I}}^{\omega}$ such that $\lambda(xu_1, q) = \lambda(yu_2, q)$. But then $\lambda(vxu_1, q_0) = \lambda(vyu_2, q_0)$; therefore, f is not injective. $\square$

Thus, to establish the injectiveness of the mapping, it suffices to find automata that recognize the sets $U_{x,q}$ for every $q \in Q$; then, one should determine the pairwise intersections of the rational sets defined by these automata and analyze whether these sets are empty.

It follows from the above that there exists an algorithm that determines by a finite automaton whether the corresponding transformation is injective and surjective.

Let us describe the algorithm that constructs, by a finite automaton defining an invertible mapping, an automaton that defines the inverse mapping.

Let $A_{q_0} = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q, \lambda, \pi\rangle_{q_0}$ be an accessible automaton and f be the invertible mapping defined by this automaton. We will also consider the mappings $f_q\colon X_{\mathrm{I}}^{\omega} \to X_{\mathrm{O}}^{\omega}$ defined by the equality $f_q(u) = \lambda(u, q)$. Since f is invertible, all mappings f_q are injective.

Let $w \in X_{\mathrm{O}}^{*}$ and $q \in Q$. Then, a word $u \in X_{\mathrm{I}}^{\omega}$ lies in $f_q^{-1}(wX_{\mathrm{O}}^{\omega})$ if and only if $f(vu) \in \lambda(v, q_0)wX_{\mathrm{O}}^{\omega}$, where $v \in X_{\mathrm{I}}^{*}$ is an arbitrary word such that $\pi(v, q_0) = q$. The set $\lambda(v, q_0)wX_{\mathrm{O}}^{\omega}$ is clopen, and f is a continuous mapping; therefore, $f^{-1}(\lambda(v, q_0)wX_{\mathrm{O}}^{\omega})$ is also clopen.

Thus, the following relation holds:

$$vf_q^{-1}(wX_{\mathrm{O}}^{\omega}) = vX_{\mathrm{I}}^{\omega} \cap f^{-1}(\lambda(v, q_0)wX_{\mathrm{O}}^{\omega}),$$

which implies that the set $f_q^{-1}(wX_{\mathrm{O}}^{\omega})$ is also clopen and therefore can be represented as a finite disjoint union $u_1X_{\mathrm{I}}^{\omega}\cup u_2X_{\mathrm{I}}^{\omega}\cup\ldots\cup u_mX_{\mathrm{I}}^{\omega}$. The words $u_1,u_2,\ldots,u_m$ can be found in a finite number of steps by enumerating the words from X_{I}^* in the increasing order of their lengths. As soon as a word $u\in X_{\mathrm{I}}^*$ such that $\lambda(u,q)$ is not a prefix of the word w is found, we can eliminate the words from X_{I}^* that begin with u. Similarly, as soon as a minimal word u is found such that $\lambda(u,q)$ begins with w, we eliminate the words beginning with u and include u in the list of sought-for words $u_1,u_2,\ldots,u_m$. It can be readily proved that this algorithm stops in a finite number of steps.

If $f_q^{-1}(wX_{\mathrm{O}}^{\omega})=u_1X_{\mathrm{I}}^{\omega}\cup u_2X_{\mathrm{I}}^{\omega}\cup\ldots\cup u_mX_{\mathrm{I}}^{\omega}$, then

$$\mathcal{P}(f_q^{-1}(wX_{\mathrm{O}}^{\omega}))=\mathcal{P}\{u_1,u_2,\ldots,u_m\}.$$

Thus, one can efficiently determine $\mathcal{P}(f_q^{-1}(wX_{\mathrm{O}}^{\omega}))$. Denote $\mathcal{P}(f_q^{-1}(wX_{\mathrm{O}}^{\omega}))=L_q(w)$.

Lemma 2.20. *Let $u_0,v\in X_{\mathrm{O}}^*$, $q\in Q$, and $u=L_q(u_0)$, where u_0 is a prefix of the word $\lambda(u,q)v$. Then, the following equality holds*:

$$L_q(\lambda(u,q)v)=uL_{\pi(u,q)}(v).$$

Proof. In order that the word $\lambda(uw,q)$ begin with $\lambda(u,q)v$, it is necessary and sufficient that $\lambda(w,\pi(u,q))$ begin with v. In addition, all words from $f_q^{-1}(\lambda(u,q)vX_{\mathrm{O}}^{\omega})\subseteq f_q^{-1}(u_0X_{\mathrm{O}}^{\omega})$ begin with $u=L_q(u_0)$.

Therefore, $f_q^{-1}(\lambda(u,q)vX_{\mathrm{O}}^{\omega})=uf_{\pi(u,q)}^{-1}(vX_{\mathrm{O}}^{\omega})$; hence, $L_q(\lambda(u,q)v)=uL_{\pi(u,q)}(v)$. □

Note that, if w_n is the beginning, of length n, of an infinite word $w\in X_{\mathrm{O}}^{\omega}$, then $\lambda(L_q(w_n),q)$ also is the beginning of the word w, and $|\lambda(L_q(w_n),q)|\to\infty$ as $n\to\infty$. In addition, if w' is the beginning of the word w'', then $L_q(w')$ is the beginning of the word $L_q(w'')$.

For every $q\in Q$, let us find the set of all words $w\in X_{\mathrm{O}}^*$ such that $L_q(w)=\varnothing$. There is a finite number of such words, and they can be found in a finite number of steps by enumerating the words from X_{O}^* in the order of increasing lengths. Then, the inverse automaton is determined as follows.

Proposition 2.21. *Let, for every state $q\in Q$ of the automaton $A_{q_0}=\langle X_{\mathrm{I}},X_{\mathrm{O}},Q,\lambda,\pi\rangle_{q_0}$, $\{w_1,w_2,\ldots,w_{m_q}\}$ be a set of words such that $L_q(w_i)=\varnothing$. Let*

$$Q'=\bigcup_{q\in Q}\{(w_1,q),(w_2,q),\ldots,(w_{m_q},q)\}.$$

For arbitrary $(w_i,q)\in Q'$ and $x\in X_{\mathrm{O}}$, we set

$$\lambda'(x,(w_i,q))=L_q(w_ix),$$
$$\pi'(x,(w_i,q))=\Big(w_ix-\lambda(L_q(w_ix),q),\ \pi(L_q(w_ix),q)\Big).$$

Then, the finite asynchronous initial automaton

$$A'_{(\varnothing,q_0)}=\langle X_{\mathrm{O}},X_{\mathrm{I}},Q',\pi',\lambda'\rangle_{(\varnothing,q_0)}$$

defines the mapping inverse to that defined by the automaton A_{q_0}.

Proof. Let us prove that the transition function is well defined. Applying Lemma 2.20 for $u_0=w_ix$ and $v=w_ix-\lambda(L_q(w_ix),q)$, we obtain

$$L_q(w_ix)=L_q(\lambda(u,q)v)=L_q(w_ix)L_{\pi(u,q)}(v);$$

hence, $L_{\pi(u,q)}(v)=\varnothing$ and $\pi'(x,(w_i,q))\in Q'$.

Using a simultaneous induction on the length of word w, let us prove that $\lambda'(w,(\varnothing,q_0)) = L_{q_0}(w)$ and that the equality $\pi'(w,(\varnothing,q_0)) = (w_i,q)$ implies the equalities

$$\pi(L_{q_0}(w),q_0) = q,$$
$$\lambda(L_{q_0}(w),q_0)w_i = w.$$

For the word $w = \varnothing$, this is true. Suppose that we have proved our assertion for all words of length n. Let wx be a word of length $n+1$, where $x \in X_{\mathrm{O}}$. Let $\pi'(w,(\varnothing,q_0)) = (w_i,q)$. From the induction hypothesis and Lemma 2.20 (for $u_0 = w$ and $v = w_i x$), it follows that

$$L_{q_0}(wx) = L_{q_0}(\lambda(L_{q_0}(w),q_0)w_i x) = L_{q_0}(w)L_q(w_i x);$$

hence, $L_{q_0}(wx) = \lambda'(wx,(q_0,\varnothing))$.

By definition,

$$\pi'(wx,(\varnothing,q_0)) = \pi'(x,(w_i,q)) = \Big(w_i x - \lambda(L_q(w_i x),q),\ \pi(L_q(w_i x),q)\Big);$$

however,

$$\pi(L_{q_0}(wx),q_0) = \pi(L_{q_0}(w)L_q(w_i x),q_0) = \pi(L_q(w_i x),q),$$

while

$$\begin{aligned}\lambda(L_{q_0}(wx),q_0)(w_i x - \lambda(L_q(w_i x),q)) &= \lambda(L_{q_0}(w),q_0)\lambda(L_q(w_i x),q)(w_i x - \lambda(L_q(w_i x),q))\\ &= \lambda(L_{q_0}(w),q_0)w_i x = wx.\end{aligned}$$

The inductive proof is completed.

Let $u \in X_{\mathrm{O}}^{\omega}$ be an arbitrary infinite word and u_n, $n \in \mathbb{N}$, be its beginning of length n. Then, the sequence $\lambda(\lambda'(u_n,(\varnothing,q_0))) = \lambda(L_{q_0}(u_n),q_0)$, $n = 1,2,\dots$, tends to u as $n \to \infty$; therefore, the automaton $A'_{(\varnothing,q_0)}$ is the inverse of the automaton A_{q_0}. $\square$

Thus, there exist an algorithm for determining the composition of two automata, an algorithm that determines whether the mapping defined by a finite automaton on infinite words is invertible, and an algorithm for determining an inverse automaton. Moreover, by Theorem 2.15, there exists an algorithm that determines whether two given automata define identical mappings. Thus, we have the following theorem.

Theorem 2.22. *In the group $\mathcal{Q}$ and semigroup $\mathcal{F}$, the problem of equality of words is solvable. In particular, this problem is solvable in an arbitrary finitely generated group that is embeddable in $\mathcal{Q}$.*

Since $\mathcal{Q}$ and $\mathcal{F}$ are infinitely generated objects, we emphasize that the solvability of the word problem in these objects means the existence of an algorithm that recognizes, by any two compositions of several finite automata, whether or not they are ω-equivalent.

3. SYNCHRONOUS AUTOMATA

3.1. The Mealy automata. Recall that an automaton is called synchronous if, for any $x \in X_{\mathrm{I}}$ and $q \in Q$, the word $\lambda(x,q)$ consists of a single letter. The synchronous automata are also referred to as the *Mealy automata.* A synchronous automaton is called the *Moore automaton* if the output function can be represented as $\lambda(x,q) = \mu(\pi(x,q))$, where $\mu\colon Q \to X_{\mathrm{O}}$ is a certain mapping. For every Mealy automaton, there exists an equivalent Moore automaton.

Below, we will consider the automata whose input and output alphabets coincide.

It follows from the definition of composition of two automata that the composition of two synchronous automata is a synchronous automaton. Therefore, all mappings $f\colon X^\omega \to X^\omega$ defined by synchronous automata form a semigroup. We denote this semigroup by $\mathcal{SA}(X)$. All transformations defined by finite synchronous automata also form a semigroup, which we denote by $\mathcal{SF}(X)$. Note that, in contrast to the asynchronous case, the semigroups $\mathcal{SA}(X)$ and $\mathcal{SF}(X)$ depend on the alphabet X.

Each state $q \in Q$ of a synchronous automaton specifies a certain mapping $\lambda_q\colon X_\mathrm{I} \to X_\mathrm{O}$ defined by the formula $\lambda_q(x) = \lambda(x, q)$. For this reason, the arrows in the graph of a synchronous automaton are sometimes labeled by a single letter x rather that by the double label of the form $x|y$, and every vertex q is simultaneously labeled by the mapping λ_q. It is these diagrams that we will mainly use in the case of a synchronous automaton.

By definition, a synchronous automaton is always nondegenerate. Moreover, a synchronous automaton preserves the lengths of finite words and its action on the space X^ω uniquely defines the action on X^*, and vice versa. In particular, the semigroups of transformations defined by (finite) synchronous automata on the sets X^ω and X^* are isomorphic as abstract semigroups, and the concepts of equivalence and ω-equivalence coincide for synchronous automata.

In addition, each class of equivalent initial synchronous automata contains a unique accessible synchronous automaton without different equivalent states. Therefore, in the synchronous case, the problem of existence of the states with incomplete response is not raised, and by *reduced* (or *minimal*) automaton one means the automaton without inaccessible states and pairs of equivalent states. The minimal automaton equivalent to a given one is found by simple identification of equivalent states, as is described in many monographs, for example, in [18]. Such an automaton is unique up to an isomorphism and has the minimum number of states as compared with all synchronous automata that are equivalent to the given one.

3.2. Synchronous automatic transformations and their properties. A transformation $g\colon X^* \to X^*$ ($g\colon X^\omega \to X^\omega$) is called *synchronous automatic* if there exists a synchronous initial automaton that defines this transformation.

A transformation $f\colon X^* \to X^*$ ($f\colon X^\omega \to X^\omega$) is said to preserve the common beginnings of words if, for arbitrary $u, v \in X^*$ ($u, v \in X^\omega$), the length $|\mathcal{P}\{f(u), f(v)\}|$ of the common beginning of the words $f(u)$ and $f(v)$ is no less than the length of the common beginning of the words u and v.

The synchronous automatic transformations are described as follows.

Proposition 3.1. *A transformation $f\colon X^\omega \to X^\omega$ ($f\colon X^* \to X^*$) is synchronous automatic if and only if it preserves the common beginnings (and lengths) of words.*

Since the synchronous automatic transformations preserve the common beginnings and lengths of words, it is more convenient to define the restriction of a synchronous automatic transformation as follows.

Definition 3.1. Let $f\colon X^\omega \to X^\omega$ be a synchronous automatic transformation and $w \in X^*$ be a finite word. The *restriction of the mapping f in the word w* is the mapping $f|_w\colon X^\omega \to X^\omega$ defined by the equality

$$f(wu) = vf|_w(u),$$

where v is a finite word with the length equal to that of w.

Note that v is a common beginning of all words of the form $f(wu)$, $u \in X^\omega$, is uniquely defined, and is the image of w under the action of the synchronous automaton that defines f.

The following simple assertions show a relation between the restrictions of synchronous automatic transformations and the states of synchronous automata [89, 90].

Proposition 3.2. *If A_q defines a synchronous automatic transformation f, then $A_{\pi(w,q)}$ defines the transformation $f|_w$.*

The states of the minimal automaton that defines a synchronous automatic transformation f are in one-to-one correspondence with the restrictions $f|_w$.

The invertibility condition and the algorithm for determining the inverse of the transformation defined by a synchronous automaton are formulated in a simpler form than in the case of an asynchronous automaton.

Proposition 3.3. *Let $A = \langle X, X, Q, \lambda, \pi \rangle$ be a synchronous automaton. The following conditions are equivalent.*

1. *Automaton A_{q0} acts by invertible transformation on X^*.*
2. *Automaton A_{q0} acts by invertible transformation on X^ω.*
3. *For any accessible state $q \in Q$, the transformation $\lambda_q \colon X \to X$ is invertible.*

A noninitial automaton in which all transformations λ_q are invertible is called *invertible.*

The automaton that defines the inverse mapping is determined by the following proposition.

Proposition 3.4. *Let $A = \langle X, X, Q, \lambda, \pi \rangle$ be a synchronous invertible automaton and let $A' = \langle X, X, Q, \lambda', \pi' \rangle$, where $\pi(x,q) = \pi'(\lambda(x,q),q)$ and $\lambda(\lambda'(x,q),q) = x$. Then, for any state $q_0 \in Q$, the initial automaton A'_{q0} defines the mapping (of the set X^* or X^ω) inverse to the mapping defined by the initial automaton A_{q0}.*

The noninitial automaton A' described in the proposition is called *inverse to the automaton A.* Accordingly, the initial automaton A'_q is called *inverse to the initial automaton A_q.*

Thus, in order to determine the diagram of the automaton inverse to a given one, one should merely change the label $x|y$ to $y|x$ on every arrow.

Hence, the set of all bijective transformations defined by synchronous automata is a group with respect to the composition. This group is called a *group of synchronous automata* and is denoted by $\mathcal{GA}(X)$. The set of all bijective transformations defined by finite synchronous automata also is a group, which is called a *group of finite synchronous automata* and denoted by $\mathcal{FGA}(X)$. These groups depend on the alphabet X. Apparently, the group $\mathcal{FGA}(X)$ was first defined in [33].

Proposition 3.1 implies the following assertion.

Proposition 3.5. *For any $n \in \mathbb{N}$, a subgroup $\mathrm{St}(n) \le \mathcal{GA}(X)$ of all automata acting trivially on the set X^n is a normal subgroup of the group $\mathcal{GA}(X)$. In this case, the sequence $\mathrm{St}(1) \ge \mathrm{St}(2) \ge \mathrm{St}(3) \ge \dots$ is the base of the neighborhoods of unity in a profinite topology on the group $\mathcal{GA}(X)$.*

Thus, the group $\mathcal{GA}(X)$ is profinite, and all its subgroups, in particular the group $\mathcal{FGA}$, are residually finite.

The subgroup $\mathrm{St}(n)$ in Proposition 3.5 is called a *stabilizer of the nth level.* The stabilizer $\mathrm{St}_G(n)$ of an arbitrary subgroup $G < \mathcal{GA}(X)$ is defined similarly.

In view of Proposition 3.1, it is natural to interpret the synchronous automatic transformations as the morphisms of a certain rooted tree $T(X)$. The vertices of the tree $T(X)$ are the elements of the set X^*; two vertices u and v are incident if and only if $u = vx$ or $v = ux$ for a certain $x \in X$. The vertex $\varnothing$ is the root of the tree $T(X)$. The *morphism* of a rooted tree is the mapping that preserves the root and the incidence relation of vertices (if two vertices are connected by an edge, their images must be different).

Proposition 3.6. *A transformation* $f\colon X^* \to X^*$ *is synchronous automatic if and only if it induces the morphism of a tree* $T(X)$.

Invertible synchronous automatic transformations are in bijective correspondence with the automorphisms of a tree $T(X)$.

More detailed information about actions on trees can be found in Section 6.

The set X^ω is naturally identified with the boundary (i.e., with the set of infinite paths without repetitions that connect the root vertex with infinity) of the tree $T(X)$.

Let $\overline{\lambda} = (\lambda_n)$ be a strictly decreasing sequence of positive numbers that tends to zero and $d(\cdot,\cdot)$ be the corresponding ultrametric on the set X^ω defined in Subsection 2.1. The following refinement of Theorem 2.4 is valid.

Proposition 3.7. *A transformation* $f\colon X^\omega \to X^\omega$ *is synchronous automatic if and only if it is nonexpansive, i.e., if the inequality*

$$d(f(u), f(v)) \le d(u,v)$$

holds for any $u, v \in X^\omega$.

An invertible mapping $f\colon X^\omega \to X^\omega$ *is synchronous automatic if and only if it is an isometry of the metric space* (X^ω, d).

3.3. Wreath products and groups of automata. The groups and semigroups of automata are closely related to the wreath products. Recall that the wreath product over a finite or infinite sequence of semigroups of transformations

$$(H_1, X_1),\ (H_2, X_2),\ (H_3, X_3), \dots$$

is a semigroup $\wr_i H_i$ of all transformations g of the Cartesian product $\mathbf{X}^\omega = \prod_i X_i$ that satisfy the following conditions (the action of g on the sequence is indicated by the right superscript):

(i) if $(y_1, y_2, \dots) = (x_1, x_2, \dots)^g$, then y_i depends only on i first coordinates $x_1, \dots, x_i$ of the preimage;

(ii) if we fix $x_1^0, \dots, x_{i-1}^0$, then the transformation g_i of the set X_i defined by the equality $(x_1^0, \dots, x_{i-1}^0, x_i, \dots)^g = (y_1^0, \dots, y_{i-1}^0, x_i^{g_i}, \dots)$ belongs to the semigroup H_i.

It follows from the definitions that any transformation $g \in \wr_i H_i$ defines a sequence $g_1, g_2, \dots$, where g_i is a function on $X_1 \times \dots \times X_{i-1}$ with the values in H_i. In other words, this type of transformations can be defined by finite or infinite corteges (which are also referred to as tableaux)

$$g = [g_1, g_2(x_1), g_3(x_1, x_2), \dots], \tag{4}$$

where $g_1 \in H_1$ and $g_i(x_1, \dots, x_{i-1}) \in H_i^{X_1 \times \dots \times X_{i-1}}$. Tableau (4) acts on an element $\overline{a} = (a_1, a_2, \dots)$ by the formula

$$\overline{a}^g = (a_1^{g_1}, a_2^{g_2(a_1)}, a_3^{g_3(a_1,a_2)}, \dots).$$

Note the following simple properties of the wreath product of semigroups of transformations.[4]

- The wreath product $\wr_i H_i$ is a group if and only if each semigroup H_i is a group.
- If H_i° is a group of invertible elements of the semigroup H_i, then the group of invertible elements of the wreath product $\wr_i H_i$ coincides with the wreath product over the sequence

$$(H_1^\circ, X_1),\ (H_2^\circ, X_2), \ldots.$$

- The wreath product $\wr_i H_i$ acts faithfully both on the set $\mathbf{X}^\omega = \prod_i X_i$ and on the set $\mathbf{X}^* = \bigcup_i (X_1 \times \ldots \times X_i)$. In this case, the action on $\mathbf{X}^\omega$ is transitive if and only if each semigroup H_i acts transitively on the set X_i, while the action on $\mathbf{X}^*$ is never transitive since the sets of the form $X_1 \times \ldots \times X_i$ are invariant.

Proposition 3.8. *The semigroup $(\mathcal{SA}(X), X^\omega)$ is isomorphic (as a semigroup of transformations) to the wreath product over the infinite sequence of full symmetric semigroups of transformations of the alphabet X.*

The group of permutations $(\mathcal{GA}(X), X^\omega)$ is isomorphic (as a group of permutations) to the wreath product over the infinite sequence of symmetric groups over the alphabet X.

In particular, this implies that the group $\mathcal{GA}(X)$ is a semidirect product of the symmetric group $\mathrm{Sym}(X)$ identified with the group of automata that preserve all letters of a word except, possibly, the first letter, and the direct product $\mathcal{GA}(X)^X = \mathrm{St}(1)$, which is a normal semigroup of automata acting trivially on one-letter words. Here, $\mathrm{Sym}(X)$ acts on $\mathcal{GA}(X)^X$ by permutations of direct factors.

Let us arrange the elements of the alphabet X in the sequence $\{a_1, a_2, \ldots, a_n\}$. Then, using the above decomposition of the group $\mathcal{GA}(X)$ into a semidirect product of its subgroups, we will sometimes represent its elements as $(g_1, g_2, \ldots, g_n)\alpha$, where $(g_1, g_2, \ldots, g_n) \in \mathcal{GA}(X)^X$, $g_i \in \mathcal{GA}(X)$, and $\alpha \in \mathrm{Sym}(X)$. Here, the automaton $g = (g_1, g_2, \ldots, g_n)\alpha$ acts on the words over the alphabet X by the rule

$$(a_i x_2 x_3 \ldots)^g = a_i^\alpha (x_2 x_3 \ldots)^{g_i}.$$

Note that, if a synchronous automaton A_{q_0} admits the decomposition $(g_1, g_2, \ldots, g_n)\alpha$, then g_i are defined by the automata $A_{\pi(a_i, q_0)}$, while the permutation α coincides with the permutation $\lambda_{q_0} = \lambda(\,\cdot\,, q_0)$, where π and λ are, respectively, the transition and output functions of the automaton A_{q_0}.

Therefore, the action of any finite synchronous automaton can be uniquely defined by the following recurrent formulas:

$$g_1 = (h_{11}, h_{12}, \ldots, h_{1n})\tau_1,$$

$$g_2 = (h_{21}, h_{22}, \ldots, h_{2n})\tau_2,$$

$$\cdots$$

$$g_n = (h_{n1}, h_{n2}, \ldots, h_{nn})\tau_n,$$

where each transformation h_{ij} is equal to a certain g_i. Conversely, any set of formulas of this type uniquely defines transformations defined by a certain finite synchronous automaton.

The above formulas uniquely define certain synchronous automatic transformations also in the case when h_{ij} are arbitrary group words of g_i. Such synchronous automatic transformations are called *functionally recursive* (see [58]). The set of all functionally recursive transformations is a countable group that contains the group of invertible finite automata and does not coincide with the latter.

[4] Note that the notation $A \wr B$ used in our paper corresponds to $B \wr A$ in Western literature.

4. EXAMPLES OF SYNCHRONOUS AUTOMATON GROUPS

It follows from Proposition 3.5 that all subgroups of the group of synchronous automata $\mathcal{GA}(X)$ and all subsemigroups of the semigroup of synchronous automata $\mathcal{SA}$ are residually finite.

The language of finite synchronous automata proved to be efficient for constructing and analyzing the examples of groups related to the unbounded Burnside problem, the groups of intermediate growth, and groups with other interesting properties.

When calculating in the group generated by finitely automatic transformations, it is often convenient to pass to the group generated by all initial automata that are obtained from a given automaton by choosing different initial states. Therefore, it seems natural to study the groups *defined* by synchronous automata.

Definition 4.1. Let $A = \langle X, X, Q, \pi, \lambda\rangle$ be a synchronous automaton. A *semigroup defined by the automaton* A is a subsemigroup of the semigroup $\mathcal{SA}(X)$ that is generated by all initial automata of the form A_q, $q \in Q$.

If the automaton A is invertible, then the subgroup of the group $\mathcal{GA}(X)$ generated by the initial automata A_q for all $q \in Q$ is called the *group defined by invertible automaton.*

We denote the semigroup and group defined by the automaton A by $S(A)$ and $G(A)$, respectively.

4.1. Two-state automata. An arbitrary synchronous single-state automaton A is uniquely determined by the permutation σ that is defined by this automaton on the set of one-letter words. The action of this automaton on other words is defined by the equality $(x_1x_2\ldots)^A = x_1^\sigma x_2^\sigma \ldots$.

Let us study which groups are defined by a two-state automaton over a two-letter alphabet.

Proposition 4.1 [30]. *Let A be a two-state automaton over the alphabet $\{0,1\}$ and G be a group defined by this automaton. Then, the group G is isomorphic to one of the following groups*:

(1) *a trivial group,*

(2) *a second-order group $\mathbb{Z}_2$,*

(3) *the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$,*

(4) *an infinite cyclic group $\mathbb{Z}$,*

(5) *an infinite dihedral group $\mathbb{D}_\infty$,*

(6) *the lamplighter group $\mathbb{Z} \wr \mathbb{Z}_2$.*

Proof. Let $\{q, s\}$ be the states of the automaton A. Then, the group G is generated by the elements $a = A_q$ and $b = A_s$; here, in view of the conventions of Subsection 3.3, the following relations hold:

$$a = (x_{11}, x_{12})\sigma^{i_1},$$
$$b = (x_{21}, x_{22})\sigma^{i_2},$$

where each x_{ij} is equal either to a or to b, σ is the transposition $(0,1)$, and i_1 and i_2 are equal either to zero or to unity.

If both exponents i_k are equal to zero, then a and b act trivially, and the group defined by the automaton is trivial. Assume that $i_1 = 1$.

Then, the following cases are possible.

1. Let $a = (a, a)\sigma$. In this case, $a^2 = (a^2, a^2)$; therefore, $a^2 = 1$. The transformation a changes each letter in each word to its opposite.

If $b = (b,b)$, $b = (a,a)\sigma$, or $b = (b,b)\sigma$, then the group G has the order 2 (in the first case, $b = 1$, while, in the second and third, $b = a$).

Since the conjugation of the transformation $c = (c_1, c_2)\sigma^i$ by means of $a = (a,a)\sigma$ yields $c^a = (c_2^a, c_1^a)\sigma^i$, it remains to consider only the cases $b = (a,a)$, $b = (a,b)\sigma$, and $b = (a,b)$.

If $b = (a,a)$, then b changes, in each word, all letters except the first one; therefore, the group G is isomorphic to the Klein group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

If $b = (a,b)\sigma$, then $a^{-1}b = \sigma(a^{-1}, a^{-1})(a,b)\sigma = (a^{-1}b, 1)$; therefore, $a^{-1}b = 1$ and, hence, $a = b$, and the group G has the order 2.

If $b = (a,b)$, then $b^{-1}a = (a^{-1}, b^{-1})(a,a)\sigma = (1, b^{-1}a)\sigma$; i.e., $b^{-1}a$ acts on infinite sequences over the alphabet $\{0,1\}$ by adding unity to a 2-adic number (the so-called *adding machine*). On the other hand, a acts on 2-adic expressions by the mapping $x \mapsto -1 - x$ since it changes all digits. Therefore, the group generated by a and b is isomorphic to the group of functions of the form $(-1)^k x + n$, $n, k \in \mathbb{Z}$, with respect to the composition, i.e., to the infinite dihedral group $\mathbb{D}_\infty$.

2. Now, let $a = (b,a)\sigma$ (the case $a = (a,b)\sigma$ is considered similarly).

If $b = (b,a)\sigma$, then $a = b$ and $a = (a,a)\sigma$; i.e., $|G| = 2$.

If $b = (a,b)\sigma$, then

$$a^{-1}b = \sigma(b^{-1}, a^{-1})(a,b)\sigma = (a^{-1}b, b^{-1}a),$$
$$b^{-1}a = \sigma(a^{-1}, b^{-1})(b,a)\sigma = (b^{-1}a, a^{-1}b);$$

hence, $a = b$, and again $a = (a,a)\sigma$ and $|G| = 2$.

The case $b = (b,b)\sigma$ has already been analyzed. If $b = (a,a)\sigma$, then, as above, we prove that $b = a = (a,a)\sigma$.

If $b = (b,b)$, then $b = 1$ and $a = (1,a)\sigma$; i.e., we have again the adding machine; therefore, $G \simeq \mathbb{Z}$.

If $b = (a,a)$, then the following equalities hold:

$$ab = (b,a)\sigma(a,a) = (ba, a^2)\sigma,$$
$$ba = (a,a)(b,a)\sigma = (ab, a^2)\sigma,$$

whence we obtain $ab = ba$. Hence,

$$a^2 b = (b,a)\sigma(b,a)\sigma(a,a) = (baa, aba) = (a^2 b, a^2 b);$$

therefore, $a^2 b = 1$, and the group G is cyclic. To prove that this group is infinite, it suffices to show, for example, that a is spherically transitive, i.e., that, for any n, the number of words w of length n with $\pi(w,a) = a$ is odd (see Definition 4.2 and Proposition 4.3). The latter is easily proved by induction.

Consider the most complicated case when $b = (b,a)$. The group G is generated by the elements b and $c = b^{-1}a$.

Let us identify the alphabet $X = \{0,1\}$ with the cyclic group $\mathbb{Z}_2$. We can readily verify that c acts on infinite sequences according to the rule

$$(x_1 x_2 \ldots)^c = (x_1 + 1)x_2 x_3 x_4 \ldots.$$

The automaton b acts by the rule

$$(x_1 x_2 \ldots)^b = x_1(x_2 + x_1)(x_3 + x_2)(x_4 + x_3) \ldots.$$

This can be verified either directly or by the linearity of the automaton, which will be done in Subsection 4.6.

Let us assign every sequence $x_1x_2\dots$ the formal power series $x_1 + x_2t + x_3t^2 + \dots \in \mathbb{Z}_2[[t]]$. Then, it directly follows from the above formulas that this identification conjugates c with the mapping $\phi_c\colon F(t) \mapsto F(t)+1$, and b with $\phi_b\colon F(t) \mapsto (1+t)F(t)$. Therefore, the group G is isomorphic to the group generated by these transformations of the ring $\mathbb{Z}_2[[t]]$. This group consists of transformations of the form

$$F(t) \mapsto (1+t)^n F(t) + \sum_{s=-\infty}^{+\infty} k_s(1+t)^s, \tag{5}$$

where $n \in \mathbb{Z}$, and all coefficients $k_s \in \mathbb{Z}_2$ except, possibly, a finite number, are equal to zero. Indeed, transformations of this type form a group containing ϕ_c and ϕ_b; however, on the other hand,

$$F(t) + (1+t)^s = (F(t)(1+t)^{-s}+1)(1+t)^s = \phi_b^s \cdot \phi_c \cdot \phi_b^{-s}(F(t));$$

therefore, all transformations of type (5) belong to the group generated by ϕ_c and ϕ_b.

This implies that G is isomorphic to the lamplighter group $\mathbb{Z} \wr \mathbb{Z}_2$, where the base of the wreath product $\mathbb{Z}_2^{\mathbb{Z}}$ is identified with the normal subgroup of transformations of the form

$$F(t) \mapsto F(t) + \sum_{s=-\infty}^{+\infty} k_s(1+t)^s$$

onto which ϕ_b acts by conjugation as a shift:

$$(1+t)^{-1}\left((1+t)F(t) + \sum_{s=-\infty}^{+\infty} k_s(1+t)^s\right) = F(t) + \sum_{s=-\infty}^{+\infty} k_{s+1}(1+t)^s.$$

The case $b = (a,b)$ is reduced to the preceding case by the transition to $a^{-1} = (a^{-1}, b^{-1})\sigma$ and $b^{-1} = (a^{-1}, b^{-1})$ and conjugation by $d = (d,d)\sigma$ (then $(a^{-1})^d = ((b^{-1})^d, (a^{-1})^d)\sigma$ and $(b^{-1})^d = ((b^{-1})^d, (a^{-1})^d))$.

3. It remains to consider the case when $a = (b,b)\sigma$.

The cases $b = (a,b)\sigma$ and $b = (b,a)\sigma$ have already been considered. If $b = (b,b)\sigma$ or $b = (a,a)\sigma$, then $a = b = (a,a)\sigma$, and the group has the order 2.

If $b = (b,b)$, then $b = 1$ and $a = \sigma$; therefore, the group has the order 2.

Let $b = (a,b)$ (the case $b = (b,a)$ is considered analogously). Then,

$$a^2 = (b^2, b^2),$$
$$b^2 = (a^2, b^2);$$

therefore, $a^2 = b^2 = 1$, and the group G is dihedral. That this group is infinite follows from the fact that ab is spherically transitive, which is readily proved by Proposition 4.3 (see below) since

$$ab = (1, ba)\sigma,$$
$$ba = (ab, 1)\sigma.$$

It remains to analyze the last case, $b = (a,a)$. We have

$$a^2 = (b^2, b^2),$$
$$b^2 = (a^2, a^2),$$
$$ab = (ba, ba)\sigma,$$
$$ba = (ab, ab)\sigma;$$

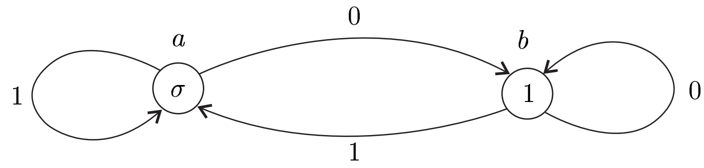


Fig. 6. Automaton defining the lamplighter group

therefore, $a^2 = b^2 = 1$ and $ab = ba = (ab, ab)\sigma$. Hence, ab also has the order 2, and G is the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This can also be proved by noting that a changes all letters of a word with odd numbers to the opposite and b changes all letters with even numbers. □

Note that the above proof of the fact that one of the groups defined by a two-state automaton is a lamplighter group is substantially shorter than the proof of a similar assertion performed in [30].

At present, it is unknown which groups are defined by three-state automata over the two-letter alphabet as well as by two-state automata with the alphabet containing more than two letters.

Consider in greater detail the automaton that defines the group $\mathbb{Z} \wr \mathbb{Z}_2$. Its diagram is shown in Fig. 6.

Proposition 4.2. *Let A be an automaton depicted in Fig. 6 and $A^{(n)}$ be its nth composition. Then, the automaton $A^{(n)}$ has 2^n pairwise nonequivalent states; for each state q of this automaton, the initial automaton $A_q^{(n)}$ is accessible.*

Proof. Consider the reduced automaton corresponding to the transformation b^n. To this end, we consider all possible restrictions $b^n|_v$. Let $v = x_1x_2 \ldots x_m$ and $g = x_1 + x_2t + \ldots + x_m t^{m-1} \in \mathbb{Z}_2[t]$ be the corresponding polynomial. Let $w = y_1y_2 \ldots \in X^\omega$ be an arbitrary infinite word, $F(t) = y_1 + y_2t + \ldots$ be the corresponding series, and $G(t)$ be the series that corresponds to the image of w under the action of $b^n|_v$. Then,

$$(1+t)^n(g + t^m F(t)) = \tilde{g} + t^m G(t),$$

where $\tilde{g} \in \mathbb{Z}_2[t]$ is a polynomial of degree at most $m-1$.

Thus, $b^n|_v$ acts on formal power series by the formula

$$F(t) \mapsto (1+t)^n F(t) + f,$$

where f is a polynomial such that $\deg((1+t)^n g - t^m f) < m$, i.e., the quotient resulting from the division of the polynomial $(1+t)^n g$ by the polynomial t^m. Since the degree of the polynomial g is at most $m-1$, the polynomial f has the degree at most $n-1$.

Let f be an arbitrary polynomial of degree at most $n-1$ and g be the quotient of the division of $t^m f$ (where $m \geq n$) by $(1+t)^n$. Then, $\deg g < m$ and $\deg((1+t)^n g - t^m f) < n \leq m$. Thus, f may be arbitrary, and the number of restrictions of b^n is equal to the number of polynomials $f \in \mathbb{Z}_2[t]$ of degree at most n, i.e., 2^n. Hence, the number of states of the automaton $A^{(n)}$ is also equal to 2^n, and all of them are pairwise nonequivalent.

Note that the quotient of the division of $(1+t)^n g$ by t^m is independent of the coefficients of the polynomial g at t^k, $k < m-n$; therefore, the restriction $b^n|_v$ depends only on the last n letters of the word v. This implies the accessibility of any state of the automaton $A^{(n)}$ from any other one. □

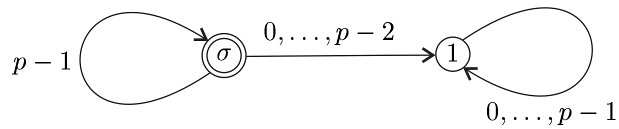


Fig. 7. Adding machine

4.2. Adding machine. The *adding machine*, or *odometer* is the transformation of the space X^ω, $X = \{0, 1, \dots, p-1\}$, defined by the initial automaton, over the alphabet X, depicted in Fig. 7 (which is also called an adding machine).

In Fig. 7, the letter σ stands for the cyclic permutation $(0, 1, \dots, p-1)$. The initial state is that situated on the left (the automaton with the other initial state acts trivially).

A recursive definition of the adding machine is as follows:

$$a = (1, 1, \dots, 1, a)\sigma.$$

In particular, when $p = 2$, we have $a = (1, a)\sigma$.

Let us identify every infinite sequence $w = x_1 x_2 \dots$ over the alphabet X with the p-adic number

$$x_1 + x_2 p + x_3 p^2 + \dots + x_{n+1} p^n + \dots .$$

This identification conjugates the adding machine with the transformation of a ring of p-adic integers that adds unity to every number (this follows from the classical sign-shift rule). That is why the term adding machine is used.

Definition 4.2. A synchronous automatic transformation $\alpha \colon X^* \to X^*$ as well as the corresponding initial automaton are called *spherically transitive* if the cyclic group $\langle \alpha \rangle$ acts transitively on the set X^n for every $n \in \mathbb{N}$.

The interpretation of the adding machine as an addition of unity in the group of p-adic numbers implies that the adding machine acts spherically transitively. Indeed, then the sets X^n are naturally identified with the groups $\mathbb{Z}/p^n\mathbb{Z}$ on which the adding machine acts by adding $1 + p^n\mathbb{Z}$, i.e., transitively.

The following result holds (see [9, 66, 67]).

Proposition 4.3. *A synchronous automatic transformation is spherically transitive if and only if it is conjugate to the adding machine in the group $\mathcal{GA}(X)$.*

In the case of a two-letter alphabet, the spherical transitiveness of the initial automaton is analyzed by the following criterion.

Lemma 4.4. *An initial synchronous automaton $A_{q_0} = \langle X, X, Q, \lambda, \pi \rangle$ over a two-letter alphabet is spherically transitive if and only if, for any $n \in \mathbb{N}$, the number of words v of length n such that $\lambda(\,\cdot\,, \pi(v, q_0)) \colon X \to X$ is a transposition is odd.*

Proof. Let us prove by induction that the automaton A_{q_0} acts transitively at the first n levels if and only if the number of words v of length $k \le n$ such that $\lambda(\,\cdot\,, \pi(v, q_0))$ is a transposition is odd.

For $n = 1$, the assertion is obvious. Suppose that it is true for n. The automaton $A_{q_0}^{(2^n)}$ acts trivially at the level n. The transitiveness of the action of the automaton A_{q_0} at this level implies that the permutation by which the automaton $A_{q_0}^{(2^n)}$ acts on the last letter of an arbitrary word of length $n+1$ is equal to the product of all permutations of the form $\lambda(\,\cdot\,, \pi(v, q_0))$, $v \in X^n$, taken in a certain order.

If there is an even number of transpositions among the permutations $\lambda(\,\cdot\,,\pi(v,q_0))$, $v\in X^n$, then $A_{q_0}^{(2^n)}$ acts trivially at the $(n+1)$th level; therefore, A_{q_0} does not act transitively at this level. If there is an odd number of transpositions, then $A_{q_0}^{(2^n)}$ changes the last letter in every word of length $n+1$; this readily implies the transitiveness of the automaton A_{q_0} at the $(n+1)$th level. $\square$

4.3. Growth of synchronous automata. Let $A=\langle X,X,Q,\lambda,\pi\rangle$ be a finite (noninitial) synchronous automaton. Let us minimize (i.e., identify equivalent states) the composition $A^{(n)}=\underbrace{A*\ldots*A}_{n}$. Denote the automaton obtained by $\overline{A^{(n)}}$. It follows from the definition of composition of automata that $|\overline{A^{(n)}}|\le |A|^n$, where $|A|$ is the number of states of the automaton A.

Definition 4.3. The *growth function of a (noninitial) automaton* A is a function $\gamma_A\colon\mathbb{N}\to\mathbb{N}$ defined by the equality

$$\gamma_A(n)=|\overline{A^{(n)}}|.$$

Similarly, if A_q is a finite initial synchronous automaton, then, minimizing the composition $A_q^{(n)}$ (i.e., rejecting the inaccessible and identifying the equivalent states), we obtain a certain automaton $\overline{A_q^{(n)}}$. The function $\gamma_{A_q}(n)=|\overline{A_q^{(n)}}|$ is called the *growth function of the initial automaton* A_{q_0}. Obviously, $\gamma_{A_q}(n)\le\gamma_A(n)$.

Let H be a semigroup (group) generated by a finite system S of generators (in the case of a group, we assume that $S=S^{-1}$). Set $\gamma_H(n)$ equal to the number of elements of H that can be represented as a product of elements of S of length no greater than n. The function $\gamma_H(n)$ is called the *growth function* of the semigroup (group) H with respect to the system of generators S.

A semigroup H (automaton) is called a semigroup (automaton) *of polynomial growth* if there exist positive numbers C and d such that $\gamma_H(n)\le Cn^d$ for all natural n. In this case, the number $d=\overline{\lim}_{n\to\infty}\frac{\ln\gamma_H(n)}{\ln n}$ is called the exponent of polynomial growth of the semigroup (automaton).

A semigroup (automaton) has an *exponential growth* if there exists a positive number C such that the following inequality holds for any natural n:

$$\gamma_H(n)\ge e^{Cn}.$$

An initial automaton has a *logarithmic growth* if there exist positive numbers C and d such that $\gamma_A(n)\le C\ln n+d$.

A semigroup or automaton is said to have an *intermediate growth* if its growth is greater than the growth of any polynomial but less than the exponential growth.

For two nondecreasing functions $\gamma_i\colon\mathbb{N}\to\mathbb{N}$, $i=1,2$, γ_1 is said to have *no greater growth order* than γ_2 (the notation is $\gamma_1\preceq\gamma_2$) if there exists a natural number C such that $\gamma_1(n)\le\gamma_2(Cn)$. Two functions γ_1 and γ_2 have *equal growth orders* if simultaneously $\gamma_1\preceq\gamma_2$ and $\gamma_2\preceq\gamma_1$. The function γ_1 has the growth order *strictly less* than the growth order of the function γ_2 if $\gamma_1\preceq\gamma_2$ but $\gamma_2\npreceq\gamma_1$.

Denote by $[\gamma]$ the equivalence class of the growth functions. It is also called the *growth order* (or the *growth degree*) of the function γ.

If γ_1 and γ_2 are the growth functions of the semigroup (group) H with respect to different systems of generators, then the growth orders of the functions γ_1 and γ_2 are equal, i.e., $[\gamma_1]=[\gamma_2]$.

Hence, the property of being a semigroup of polynomial, exponential, or intermediate growth does not depend on the choice of the system of generators S of the semigroup.

The following theorem describes all cancellative semigroups of polynomial growth.

Theorem 4.5 [74]**.** *A finitely generated cancellative semigroup H has a polynomial growth if and only if it contains a nilpotent subsemigroup of finite index.*

Recall that a semigroup is called *nilpotent* (following A.I. Mal'tsev) if, for a certain natural n and any elements $x, y, a_1, a_2, \dots, a_n$ of the semigroup H, the equality $X_n = Y_n$ holds, where X_n and Y_n are defined inductively:

$$X_0 = x, \quad Y_0 = y, \qquad X_{i+1} = X_i a_{i+1} Y_i, \quad Y_{i+1} = Y_i a_{i+1} X_i.$$

In [80], Mal'tsev proved that a group is n-step nilpotent if the law $X_n = Y_n$ holds while the law $X_{n-1} = Y_{n-1}$ does not hold in this group.

A subsemigroup $H_0 \leq H$ has a *finite index* if there exists a finite set $K \subseteq H$ such that, for every $h \in H$, there exists $k \in K$ such that $hk \in H_0$.

Theorem 4.5 is a generalization of Gromov's theorem [26] that describes the groups of polynomial growth as almost nilpotent groups.

The following property is valid that relates the concept of growth of an automaton to the concept of growth of a semigroup (see [74]).

Proposition 4.6. *The growth function of a noninitial automaton A coincides with the growth function of the semigroup $S(A)$ with respect to the system of generators $S = \{A_q\colon q \in Q\}$.*

Corollary 4.7. *An invertible automaton A has a polynomial growth if and only if the semigroup $S(A)$ contains a nilpotent subsemigroup of finite index.*

Recall that the group $\mathcal{FGA}(X)$ is residually finite; hence, for any synchronous automaton A, the group $G(A)$ and, consequently, the semigroup $S(A)$ are residually finite.

For the class of groups that are approximable by finite p-groups, there exists a lacuna in the set of growth degrees: if the growth order of such a group is strictly less than $e^{\sqrt{n}}$, then it has a polynomial growth [75]. Therefore, a similar property is also characteristic of the growth of automata.

Theorem 4.8. *Let A be an invertible synchronous automaton over the alphabet $\{0, 1, \dots, p-1\}$ (p is a prime number), where, for any state q, the output function $\lambda(\,\cdot\,, q)$ is a power of the cyclic permutation $(0, 1, \dots, p-1)$. Then, if the growth order of the automaton A is strictly less than $e^{\sqrt{n}}$, $S(A)$ contains a nilpotent subsemigroup of finite index, and A has a polynomial growth.*

Using Proposition 4.6, we can construct the automata that have polynomial, exponential, and intermediate growths.

An example of the automaton of intermediate growth is the automaton that has the diagram depicted in Fig. 14 and defines the Grigorchuk 2-group.

Indeed, all initial automata that can be obtained by choosing the initial state in the above automaton have the order 2 in the group $\mathcal{FGA}(X)$; therefore, the semigroup defined by this automaton coincides with the group that it defines. As was noted by Merzlyakov [81], this group is isomorphic to the 2-group constructed in [71]; and as was proved in [72], it is a group of intermediate growth. Moreover, the following estimates are valid: $e^{n^\alpha} \preceq \gamma(n) \preceq e^{n^\beta}$, where $\alpha = 0.51$ and $\beta = 0.7$ [72, 79, 4].

Another example of the automaton of intermediate growth is the automaton over the alphabet $X = \{0, 1, 2\}$ represented in Fig. 9; this follows from the result of Gupta and Fabrykowski [19] concerning the intermediate character of growth of the group defined by this automaton.

Moreover, there exists a two-state automaton of intermediate growth over a three-letter alphabet. Namely, it was proved in [5] that the group defined by the automaton shown in Fig. 8 has an intermediate growth.

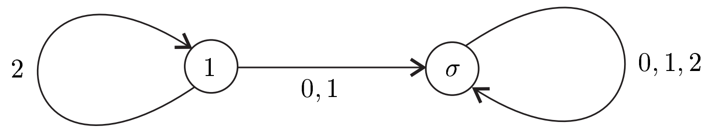


Fig. 8. A two-state automaton of intermediate growth

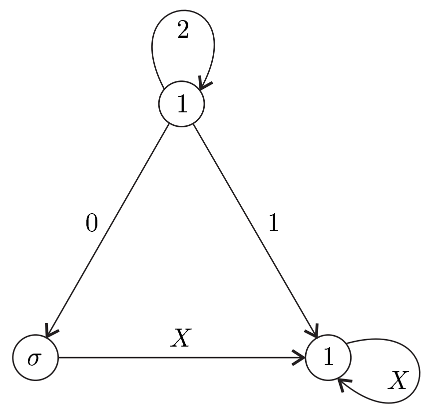


Fig. 9. A three-state automaton of intermediate growth

New examples of three-state automata of intermediate growth can be obtained from the recent results of L. Bartholdi and Yu. Leonov on the intermediate character of the growth of the Gupta–Sidki groups and other GGS-groups. For example, such is the automaton shown in Fig. 15.

An example of the automaton of polynomial growth is given by the adding machine—the automaton shown in Fig. 7. Since the semigroup defined by this automaton is isomorphic to the semigroup of nonnegative integers, its growth function is equal to $\gamma(n) = n + 1$.

Let us analyze the growth of the adding machine considered as an initial automaton A_q. For this purpose, we have to find the number of states of the reduced automaton $\overline{A_q^{(n)}}$, i.e., to find the number of different restrictions of the transformation, of the set X^ω, defined by this automaton.

Let $w \in X^*$ be a finite word. Let us interpret it as a p-adic representation ($X = \{0, 1, \dots, p-1\}$) of a certain integer m. This number belongs to the interval $0 \le m \le p^s - 1$, where s is the length of the word w. The automaton $A_q^{(n)}$ acts on the p-adic representations as addition of the number n. The definition of the restriction of automatic transformation and the addition rules for p-adic numbers imply that the restriction of $A_q^{(n)}$ in the word w is given by the transformation $A_q^{(k)}$, where $k = [\frac{m+n}{p^s}]$ (here, $[\,\cdot\,]$ is the integer part).

If $n < p^s$, then $0 = [\frac{n}{p^s}] \le k \le [\frac{n+p^s-1}{p^s}] = 1$. If $p^s \le n$, then the amount of numbers of the form $[\frac{m+n}{p^s}]$, where $0 \le m \le p^s - 1$, is at most

$$\left[\frac{n + p^s - 1}{p^s}\right] - \left[\frac{n}{p^s}\right] + 1 < \frac{n + p^s - 1}{p^s} - \frac{n}{p^s} + 2 = 3 - p^{-s} < 3.$$

Thus, the automaton $\overline{A_q^{(n)}}$ has at most $2\log_p n + 2$ states, and the adding machine considered as an initial automaton has a logarithmic growth.

The automaton A depicted in Fig. 6 defines the group $\mathbb{Z} \wr \mathbb{Z}_2$; however, as was proved in [30], $S(A)$ is isomorphic to a free semigroup with two generators; hence, this automaton has an exponential growth. Its growth, as the growth of an initial automaton, is also exponential, which follows from Proposition 4.2.

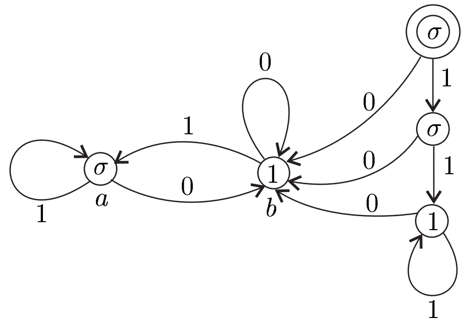


Fig. 10. A spherically transitive automaton of exponential growth

In the general case, the growth of the initial automaton may be substantially lower than its growth considered as the growth of a noninitial automaton. For example, the growth of the automaton that defines the Grigorchuk 2-group is bounded for an arbitrary choice of the initial state since its second power is a unit automaton.

No analogue of Proposition 4.6 is known for the initial automata.

The study of the growth order of an initial automaton may be useful for the analysis of finite automatic transformations since the growth is the invariant of this automaton with respect to the conjugation in the group $\mathcal{FGA}(X)$.

Proposition 4.9. *Let A_p and B_q be initial automata conjugate in the group of finite synchronous automata $\mathcal{FGA}(X)$, and $\gamma_A(n)$ and $\gamma_B(n)$ be their growth functions. Then, there exists a number $K > 1$ such that*

$$K^{-1}\gamma_A(n) \le \gamma_B(n) \le K\gamma_A(n)$$

for any $n \in \mathbb{N}$.

Proof. If C is an automaton that conjugates the automata A and B, then $C^{-1} * A^{(n)} * C = B^{(n)}$ for every n; therefore, $|\overline{B^{(n)}}| = |\overline{C^{-1} * A^{(n)} * C}| \le |C^{-1}|\,|\overline{A^{(n)}}|\,|C| = |C|^2|\overline{A^{(n)}}|$ and $|\overline{A^{(n)}}| = |\overline{C * B^{(n)} * C^{-1}}| \le |C|^2|\overline{B^{(n)}}|$; hence, it suffices to set $K = |C|^2$. □

Using Proposition 4.9, we can display an example of two automata that are conjugate in the group $\mathcal{GA}(X)$ of all synchronous automata but are not conjugate in the group $\mathcal{FGA}(X)$ of finite synchronous automata.

Consider the automaton A_q whose diagram is shown in Fig. 10. The double circle indicates the initial state.

Using Lemma 4.4, we can easily show that the automaton A_q is spherically transitive and, hence, is conjugate in the group $\mathcal{GA}(X)$ with the adding machine. Let us show that its growth is exponential, which will imply that it is not conjugate in $\mathcal{FGA}(X)$ with the adding machine since the latter has a logarithmic growth.

The part of the automaton A_q accessible from the state b is the automaton L defining $\mathbb{Z} \wr \mathbb{Z}_2$ (cf. Fig. 6). Note also that, for an arbitrary $v \in X^*$, the state $\pi(v, q)$ belongs to L except for the case when v consists only of ones.

Let us prove that the number $|\overline{A_q^{(n)}}| \ge 2^n$ for every n. Let k be such that $2^k > n$. The automaton A_q acts on the set X^k as a cyclic permutation. Let $v_1, v_2, \dots, v_{2^k}$ be an ordering of the elements of the set X^k such that $v_i^{A_q} = v_{i+1}$ (the indices are considered modulo 2^k). Then, the restriction of the action of A_q^n in the word v_i is equal to the action of the automaton

$$A_{\pi(v_i,q)}A_{\pi(v_{i+1},q)} \dots A_{\pi(v_{i+n-1},q)}.$$

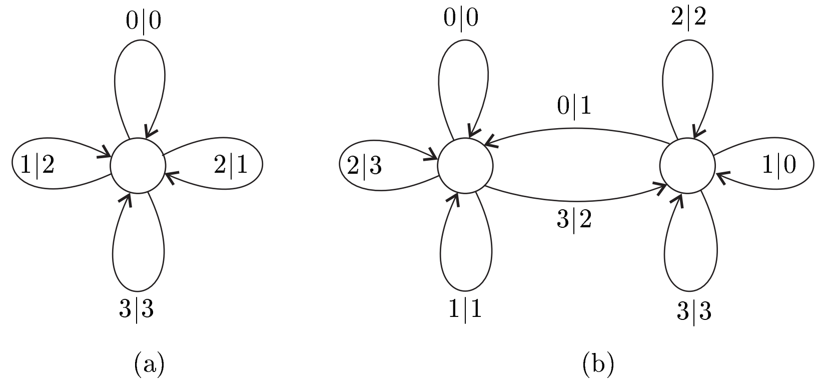


Fig. 11. Automata that generate $\mathrm{GL}(2,\mathbb{Z})$

However, only for one word v_i the automaton $A_{\pi(v_i,q)}$ is not the automaton L; therefore, there exists v_i such that the restriction of the action of $A_q^{(n)}$ in the word v_i represents a certain state of the automaton $L^{(n)}$. However, by Proposition 4.2, any of 2^n states of $L^{(n)}$ is accessible from any other its state. Therefore, $|\overline{A_q^{(n)}}| \geq 2^n$; i.e., A_q has the exponential growth.

4.4. Abelian and linear groups. The construction of the adding machine can be generalized by constructing an automaton that defines the free abelian group $\mathbb{Z}^n$ of rank n. Moreover, linear groups can also be represented by automata.

Let $\overrightarrow{m} = (m_1, m_2, \ldots, m_n)$ be an arbitrary integer 2-adic vector, where

$$m_i = a_{i0} + a_{i1}2 + a_{i2}2^2 + \ldots + a_{ik}2^k + \ldots$$

and $a_{ij} \in \{0,1\}$. Let us define a bijection ϕ between the set of n-dimensional 2-adic integer vectors and the space X^ω of infinite words over the alphabet $X = \{0,1\}^n$, whose elements are given by the corteges of the form $(x_1, x_2, \ldots, x_n)$, where $x_i \in \{0,1\}$, in the following way:

$$\phi(\overrightarrow{m}) = (a_{10}, a_{20}, \ldots, a_{n0})(a_{11}, a_{21}, \ldots, a_{n1}) \ldots (a_{1k}, a_{2k}, \ldots, a_{nk}) \ldots .$$

The following assertions were proved in [10].

Theorem 4.10. *For any integer vector* $\overrightarrow{m} = (m_1, m_2, \ldots, m_n)$ *and matrix* $A \in \mathrm{GL}(n,\mathbb{Z})$ *with integer elements, the conjugation of the transformation* $\overrightarrow{x} \mapsto \overrightarrow{x}A + \overrightarrow{m}$ *by means of* ϕ *is a finitely automatic transformation of the space* X^ω.

Thus, we obtain a representation of a free abelian group $\mathbb{Z}^n$, a linear group $\mathrm{GL}(n,\mathbb{Z})$, and an affine group by synchronous finitely automatic transformations over the alphabet of 2^n letters.

In particular, the affine group of transformations of the form $\overrightarrow{x} \mapsto \overrightarrow{x}A + \overrightarrow{m}$, where $A \in \mathrm{GL}(n,\mathbb{Z})$ and $\overrightarrow{m} \in \mathbb{Z}^2$, is defined by the automata represented in Fig. 11. The automata act on the alphabet $\{0,1,2,3\}$, where 0 corresponds to the vector $(0,0)$, 1 to the vector $(0,1)$, 2 to $(1,0)$, and 3 to $(1,1)$. The automaton in Fig. 11a corresponds to the right multiplication of vectors by the matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, while the automaton in Fig. 11b acts on the vectors by multiplication by the matrix $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ when the left state is chosen as the initial one and acts by the transformation $\overrightarrow{m} \mapsto \overrightarrow{m}T + (0,1)$ when the right state is chosen as the initial one.

On the other hand, we can define a bijection ψ between the space of n-dimensional 2-adic integer vectors and the space X^ω of infinite words over the two-letter alphabet $X = \{0,1\}$:

$$\psi(\overrightarrow{m}) = a_{10}a_{20} \ldots a_{n0}a_{11}a_{21} \ldots a_{n1} \ldots a_{1k}a_{2k} \ldots a_{nk} \ldots .$$

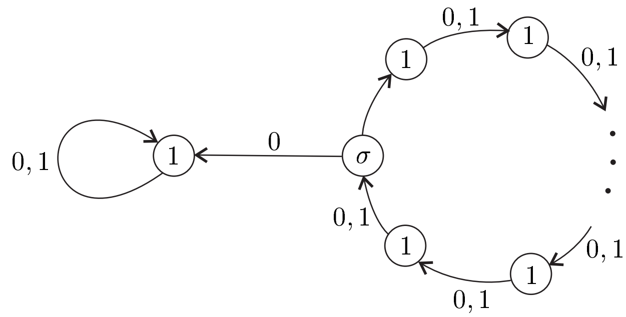


Fig. 12. An automaton that defines $\mathbb{Z}^n$

Theorem 4.11 [45]. *Let $\overrightarrow{m} = (m_1, m_2, \ldots, m_n)$ be an integral vector and $A \in \mathrm{GL}(n, \mathbb{Z})$. Then, the conjugation of the transformation $\overrightarrow{x} \mapsto \overrightarrow{x}A + \overrightarrow{m}$ by means of the bijection ψ is a synchronous finitely automatic transformation of the space X^ω if and only if all elements of the matrix A below the main diagonal are even numbers.*

Thus, the subgroup $B \leq \mathrm{GL}(n, \mathbb{Z})$ of index $(2^n - 1)(2^{n-1} - 1) \ldots (2^2 - 1)$ of matrices with even elements below the diagonal can be represented as a group of synchronous finitely automatic transformations over a two-letter alphabet.

In this case, a group of synchronous finitely automatic transformations that is isomorphic to $\mathbb{Z}^n$ and is obtained by the construction from Theorem 4.11 is defined by the automaton depicted in Fig. 12.

The methods of [10, 45] also allow one to construct the embeddings of linear groups over the rings of p-adic integers.

In [45], the subgroups of the group of synchronous automata $\mathcal{FGA}(X)$ for $|X| = 2$ were investigated that can be represented as $G(A)$ for a certain finite automaton A and are isomorphic to the free abelian groups $\mathbb{Z}^n$. It was proved that, for any n, the set of such subgroups is decomposed into a finite number of classes such that any two groups from the same class intersect over a semigroup of finite index.

In particular, for $n = 1$, there are two such classes: the class containing the group defined by the adding machine and the class containing the group defined by an automaton with recursive definition $\mu = (1, \mu^{-1})\sigma$, where $\sigma = (0, 1)$ is a transposition.

For $n = 2$, there are six such classes. In the general case, the number of classes is equal to the number of conjugation classes in $\mathrm{GL}(n, \mathbb{Z})$ of matrices with integer elements that have irreducible characteristic polynomials and the determinants equal to 2. Other examples of abelian groups of synchronous automatic transformations were constructed in [83].

The fact that linear groups can be represented by finite synchronous automata implies that there exist free subgroups in the group $\mathcal{FGA}(X)$ for any finite alphabet X, $|X| > 1$. Indeed, there exists a free subgroup in the subgroup B defined above; this follows, for example, from the Tits alternative. The group $\mathcal{FGA}(X)$ for $|X| > 2$ contains an isomorphic copy of the group $\mathcal{FGA}(\{0, 1\})$.

Aleshin's paper [64] was the first work that constructed an example of a free subgroup of the group $\mathcal{FGA}(X)$ for $|X| = 2$; in this paper, it was argued that the automata depicted in Fig. 13 generate a free group. Unfortunately, a detailed proof of this fact has not been published yet.

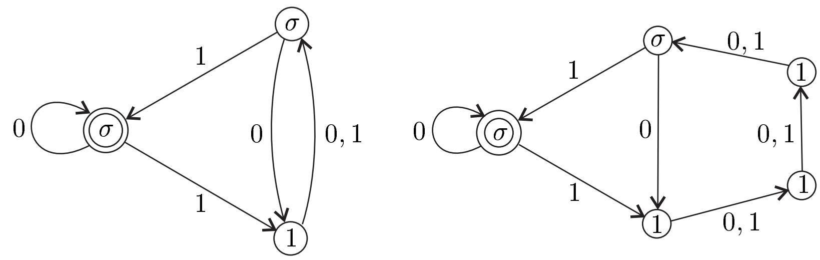


Fig. 13. The Aleshin automata

The problem, posed in [59], about whether the first Aleshin automaton defines a free group with three generators was solved positively[5].

On the other hand, using the ideas of [40], S. Mozes constructed a four-state automaton that defines a free group. In [40], it was shown that, for an arbitrary uniform lattice Γ in the group of automorphisms of an unrooted regular tree, any automorphism of the tree that commensurates Γ (i.e., automorphism g such that $\Gamma^g \cap \Gamma$ is a subgroup of finite index in Γ and Γ^g) can be interpreted as a "periodic recoloring." The latter condition can easily be interpreted as the finite automatic property. The automata obtained in this way are described in [42].

An example of a free group of synchronous finitely automatic transformations is obtained from the representation, constructed in [10], of the full linear group $\mathrm{GL}(2, \mathbb{Z})$ by finite automata over a four-letter alphabet.

The first example of a free subgroup of the group $\mathcal{FGA}(X)$ over a two-letter alphabet was constructed in [49, 82] with the use of the "marker" technique from the theory of cellular automata (the automorphisms of the Bernoulli shift, see [35]). Namely, it was proved that the free product of any number of second-order cyclic groups is a subgroup of the group $\mathcal{FGA}(X)$.

The group from [49] can be realized as a subgroup of the group of infinite unitriangular matrices over the field of two elements by the following construction.

Let $\Bbbk$ be an arbitrary finite field. Consider the set $\Bbbk^\omega$ of infinite sequences over the alphabet $\Bbbk$ as a vector space over $\Bbbk$ with coordinatewise operations of addition and multiplication by a number. An arbitrary matrix

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \dots \\ 0 & 1 & a_{23} & a_{24} & \dots \\ 0 & 0 & 1 & a_{34} & \dots \\ . & . & . & . & \dots \end{pmatrix}, \tag{6}$$

where $a_{ij} \in \Bbbk$, defines a linear transformation of the space $\Bbbk^\omega$ by the right multiplication of the row vectors. It can be readily verified that this transformation of the set of infinite words $\Bbbk^\omega$ is synchronous automatic.

Thus, we have a natural isomorphism between the group of upper unitriangular matrices over the field $\Bbbk$ and the subgroup of the group of synchronous automata over the alphabet $\Bbbk$ (see [84]).

[5] A work is in preparation: Grigorchuk, R.I. and Żuk, A., A free group defined by a three-state automaton.

For example, the action of the automaton depicted in Fig. 6 with the initial state b (the automaton that defines the lamplighter group) is defined in the space $\mathbb{Z}_2^\omega$ by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ . & . & . & . & \dots \end{pmatrix}.$$

Matrix (6) defines a finitely automatic transformation if and only if there exist natural k, l, i_0, and j_0 such that $a_{i,i+j} = a_{i+k,i+j+k}$ for any $i \geq i_0$ and $j > 0$ and $a_{i,i+j} = a_{i,i+j+l}$ for any $i > 0$ and $j \geq j_0$; i.e., all rows and all diagonals of this matrix are almost periodic with uniformly bounded lengths of periods.

Let $\Bbbk = F_2$ be a field with two elements. Define the matrices F_1 and F_2 as follows:

$$F_1 = \begin{pmatrix} E & B_1 & C_1 & O & O & \dots \\ O & E & B_1 & C_1 & O & \dots \\ O & O & E & B_1 & C_1 & \dots \\ . & . & . & . & . & \dots \end{pmatrix}, \qquad F_2 = \begin{pmatrix} E & B_2 & C_2 & O & O & \dots \\ O & E & B_2 & C_2 & O & \dots \\ O & O & E & B_2 & C_2 & \dots \\ . & . & . & . & . & \dots \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and O and E are the zero and identity matrices of the third order, respectively.

Proposition 4.12 [84]. *The group $\mathcal{F}$ generated by the matrices F_1 and F_2 is a free group of rank* 2.

Note that the matrices F_1 and F_2 satisfy the above almost-periodicity condition for the rows and diagonals; therefore, the transformations defined by these matrices are finitely automatic.

4.5. Periodic groups generated by finite automata. As was predicted by Glushkov [69], synchronous automata can be effectively used for constructing the examples of infinite finitely generated periodic groups; the question of the existence of these groups was the central point in Burnside's general problem (which was first solved in [70] with the use of the Golod–Shafarevich theorem).

The first example of infinite finitely generated periodic group generated by finitely automatic transformations was constructed in Aleshin's work in 1972 [63]. This group was generated by two initial automata over a two-letter alphabet; one automaton had three states, while the other had seven.

In 1979, Sushchanskii [86] constructed examples of infinite p-groups generated by two synchronous automatic transformations over the alphabet $\{0, 1, \dots, p-1\}$ (p is an odd prime number). The first generator is equal to $(1, \sigma, \sigma^2, \dots, \sigma^{p-1})\sigma$. To construct the second generator, we enumerate all pairs (α_i, β_i), $\alpha_i, \beta_i \in \{0, 1, 2, \dots, p-1\}$, by the numbers $1 \leq i \leq p^2$. The second generator defines the following transformation on the set of infinite words:

$$x_1 x_2 \dots \mapsto a_1(x_1) a_2(x_1, x_2) a_3(x_1, x_2, x_3) \dots,$$

where $a_n(x_1, x_2, \dots, x_n) = x_n$ always except for the following cases:

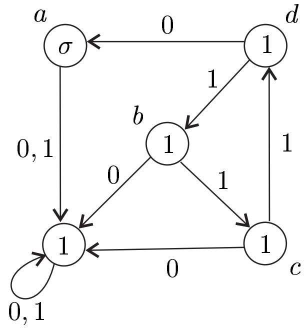


Fig. 14. Automaton defining the Grigorchuk group

(1) $a_3(2,1,x_3) = x_3 + 1$;

(2) if $n \geq 4$ and $1 \leq i \leq p^2$ are such that $n \equiv (i-3) \pmod{p^2}$ and $\beta_i \neq 0$, then

$$a_n(0,0,\dots,0,1,x_n) = x_n + 1 \qquad \text{and} \qquad a_n(1,0,0,\dots,0,1,x_n) = x_n - \alpha_i/\beta_i;$$

(3) if $\beta_i = 0$, then $a_n(1,0,0,\dots,0,1,x_n) = x_n + 1$.

We can readily show that this transformation of the space of infinite words is defined by a finite synchronous automaton.

Other constructions of groups of automata were proposed by Sushchanskii in [87, 88] in relation to the applications to the theory of factorizable groups.

In 1980, R.I. Grigorchuk constructed two examples of finitely generated 2-groups of transformations of the segment $[0,1]$ and the square $[0,1] \times [0,1]$. These groups can naturally be realized as the groups of synchronous automatic transformations over a two-letter alphabet if the points of the segment and the square are identified with binary expressions. In this interpretation, the first group is defined by the automaton depicted in Fig. 14. This group is described in more detail in [24, 31].

In 1983, N. Gupta and S. Sidki constructed the examples of p-groups ($p > 2$ is a prime number) that act on a p-regular tree by automorphisms [27, 28]. These groups are defined by a four-state automaton. The Gupta–Sidki group is generated by the cyclic permutation $\sigma = (0,1,\dots,p-1)$ and by the transformation defined by the formula $t = (\sigma, \sigma^{-1}, 1, 1, \dots, t)$.

In 1985, an automaton with the minimum number 3 of states was constructed in [73] that defines an infinite periodic group. Let us consider this example.

Let A be the automaton shown in Fig. 15. Here, $p \geq 3$ is a prime number, and $\sigma = (0,1,\dots,p-1)$ is a cyclic permutation.

Theorem 4.13. *The group $G(A)$ is an infinite p-group.*

Proof. The transformations corresponding to the initial automata obtained from the given one can be defined by the recurrent relations

$$a = (a, a, \dots, a)\sigma,$$
$$b = (a, a, \dots, a, a^2, b),$$

where $\sigma = (0,1,2,\dots,p-1)$ is a cyclic permutation of order p. In this case, $a^2 = (a^2,\dots,a^2)\sigma^2$.

Note that $a^p = (a^p, a^p, \dots, a^p)$; therefore, $a^p = 1$. Hence,

$$b^p = (a^p, \dots, a^p, a^{2p}, b^p) = (1, 1, \dots, 1, b^p);$$

therefore, $b^p = 1$.

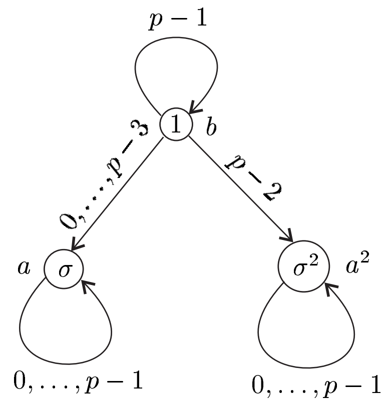


Fig. 15. Automaton defining an infinite periodic group

Thus, an arbitrary element of the group $G(A)$ can be represented as a word $g = b_{i_1} b_{i_2} \dots b_{i_n} a^{s(g)}$ in the free product of cyclic groups $\langle a\rangle * \langle b\rangle \simeq \mathbb{Z}_p * \mathbb{Z}_p$, where $b_{i_j} = a^{-i_j} b a^{i_j}$, $0 \le i_j \le p-1$, and $s\colon \langle a\rangle * \langle b\rangle \to \mathbb{Z}_p$ is a homomorphism such that $s(a) = 1$ and $s(b) = 0$. The minimum number n in this decomposition is called the length of the element g and is denoted by $l(g)$. Let us prove by induction on the length that any element g of the group $G(A)$ has the order p^m for a certain m. For $l(g) = 0$, this assertion is obvious. Suppose that we have proved it for all g such that $l(g) \le n-1$.

Let $g = b_{i_1} b_{i_2} \dots b_{i_n} a^{s(g)}$ be an element of length n. First, consider the case when $s(g) = 0$. Note that $a^{-i_k} b a^{i_k} = (a, \dots, a, a^2, a^{-i_k} b a^{i_k}, a, \dots)$, where the left-hand side is the result of application of the cyclic permutation σ^{i_k} to the cortege $(a, \dots, a, a^2, a^{-i_k} b a^{i_k})$. Hence, if $g = (g_0, g_1, \dots, g_{p-1})$, then $n = l(g) \ge l(g_0) + l(g_1) + \dots + l(g_{p-1})$. Therefore, either $l(g_i) < l(g)$ for all $0 \le i \le p-1$, and then, by the induction hypothesis, $g^{p^m} = (g_0^{p^m}, g_1^{p^m}, \dots, g_{p-1}^{p^m}) = 1$ for a certain m, or all $l(g_i)$ except one are equal to zero. However, the latter case is possible only when $g = b_{i_1}^n$, but this element always has the order p.

Now, let $g = ha^{s(g)}$, where $h = b_{i_1} b_{i_2} \dots b_{i_n}$. Then, $g^p = hh^{a^{-s(g)}} h^{a^{-2s(g)}} \dots h^{a^{-(p-1)s(g)}}$. Let $h = (h_0, h_1, \dots, h_{p-1})$. Then, $s(h_0) + s(h_1) + \dots + s(h_{p-1}) = 0$ since, for every $0 \le k \le p-1$, the sum of the values of the homomorphism s on the elements of the cortege $a^{-k} b a^k = (a, \dots, a, a^2, a^{-k} b a^k, a, \dots)$ is equal to zero. In addition, $h^{a^{-ks(g)}} = (h_{(0,k)}, h_{(1,k)}, \dots, h_{(p-1,k)})$, where (i,k) is the image of i under the action of the cyclic permutation $\sigma^{-ks(g)}$. Since p is a prime number, this implies that, if $g^p = (g_0, g_1, \dots, g_{p-1})$, then $l(g_i) = l(h) = n$ and $s(g_i) = 0$ for every $0 \le i \le p-1$. However, by the induction hypothesis, this implies that g_i has a finite order of the form p^{m_i}; consequently, g also has a finite order of the form p^m, where $m = \max m_i + 1$.

To prove that the group $G(A)$ is infinite, it is sufficient to prove that it is spherically transitive; i.e., it is sufficient to prove by induction on n that, in the set X^n of finite words of length n, there exist a word w and elements $g_0, g_1, \dots, g_{p-1} \in G(A)$ such that the first $n-1$ letters in the words w^{g_i} coincide with the first $n-1$ letters of the word w, while the last letter assumes all possible values from the alphabet. Such a word is given by 0 for $n = 1$ and the elements $1, a, a^2, \dots, a^{p-1}$, the word 00 for $n = 2$, and the word $(p-1, p-1, \dots, p-1, 0, 0)$ for the other n and elements $1, b, b^2, \dots, b^{p-1}$. $\square$

Note that, when $p = 3$, this construction leads to a group conjugate to the Gupta–Sidki group [27]. Indeed, let us conjugate the transformation a by $c = (a^2 c, ac, c)$:

$$\begin{aligned} a^c &= (c^{-1}a^{-2}, c^{-1}a^{-1}, c^{-1})(a, a, a)\sigma(a^2 c, ac, c) \\ &= (c^{-1}a^{-2}, c^{-1}a^{-1}, c^{-1})(a, a, a)(ac, c, a^2 c)\sigma = (1, 1, c^{-1}a^3 c)\sigma = \sigma \end{aligned}$$

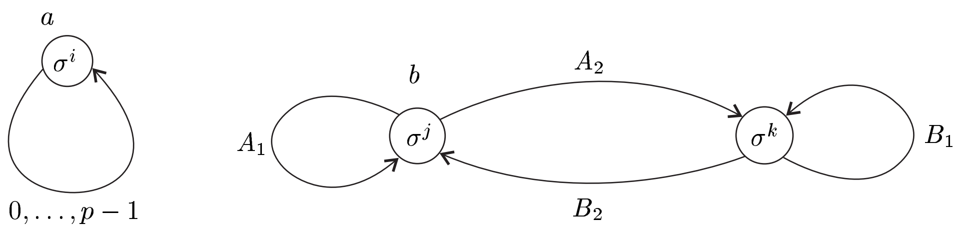


Fig. 16

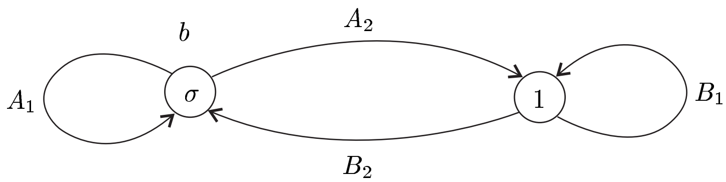


Fig. 17

since $a^3 = (a^3, a^3, a^3) = 1$. Let us conjugate b:

$$b^c = (c^{-1}a^{-2}, c^{-1}a^{-1}, c^{-1})(a, a^2, b)(a^2c, ac, c) = (a^c, (a^c)^2, b^c) = (\sigma, \sigma^2, b^c);$$

therefore, $b^c = t$.

The conjugating transformation c is finitely automatic since $ac = c\sigma$ and $a^2c = c\sigma^2$.

The group from Theorem 4.13 is generated by two initial automata with the total number of states $1 + 3 = 4$. The fact that, in the case of a p-letter alphabet, this total number is minimal for infinite periodic groups of automata is implied by the following proposition. (We emphasize that we consider automata each state of which acts on the input symbols by a power of the cyclic permutation.)

Proposition 4.14 [73]. *The groups generated by two automata with one and two states over a p-letter alphabet, where p is a prime number, cannot be infinite p-groups.*

Proof. It suffices to consider the initial automata a and b of the form shown in Fig. 16 (i, j, and k are certain integers from the interval $0 \le i, j, k \le p-1$; $A = A_1 \uplus A_2$ and $B = B_1 \uplus B_2$ are two partitions of the alphabet; the left state of the second automaton is initial).

Changing b to $b' = ba^{-l}$, where the number l, $0 \le l \le p-1$, is chosen so that $il \equiv k \pmod p$, and replacing σ and a by certain their powers, we can assume that the automata a and b have a more specific form: $a = (a, a, \ldots, a)\sigma$, while the automaton b is shown in Fig. 17.

Consider a few cases. (Note that A_2 and B_2 cannot be empty sets simultaneously.)

1. Suppose that $A_2 = \varnothing$. Then, the automaton b is equivalent to the automaton a, and $G = \langle a, b\rangle$ is a cyclic group.

2. Suppose that $A_1 = \varnothing = B_1$. Then, the automaton b acts on the input sequences by cyclically changing the coordinates on odd positions; i.e., $b = (c, c, \ldots, c)\sigma$, where $c = (b, b, \ldots, b)$.

Then,

$$ab = (a, a, \ldots, a)\sigma(c, c, \ldots, c)\sigma = (ac, ac, \ldots, ac)\sigma^2,$$
$$ba = (c, c, \ldots, c)\sigma(a, a, \ldots, a)\sigma = (ac, ac, \ldots, ac)\sigma^2;$$

hence, $ab = ba$; therefore, $G = \langle a, b\rangle$ is a commutative group.

3. Suppose that at least one of the equalities $A_1 = B_1$ or $A_2 = B_2$ is not true. Then, at least one of the intersections $A_1 \cap B_2$ or $A_2 \cap B_1$ is nonempty. Indeed, if $A_1 \cap B_2 = \varnothing$ and $A_2 \cap B_1 = \varnothing$,

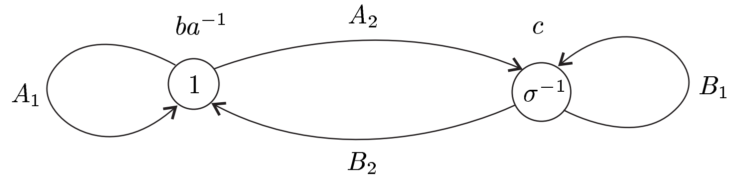


Fig. 18

then at least one of the inequalities $|A_1\cup B_2|<p$ or $|A_2\cup B_1|<p$ is valid and, hence, $|A_1|+|B_2|<p$ or $|A_2|+|B_1|<p$. Therefore, $|A_1|+|A_2|+|B_1|+|B_2|<2p$—contradiction. If $|A_1|+|B_2|=p$ and $A_1\cap B_2=\varnothing$, then $A_1=B_1$ and $A_2=B_2$.

Next, we can assume that $A_2\cap B_1$ is nonempty. Indeed, if $A_1\cap B_2\neq\varnothing$, then the automaton ba^{-1} is defined by the diagram shown in Fig. 18, and we can replace b by the automaton ba^{-1}.

If $x\in A_2$, then the automaton ba^{-1} acts on the input sequences that begin with x so that it leaves the first coordinate unchanged and acts as automaton c on the remaining part. If the order of the automaton c is unbounded, then so is the order of the automaton ba^{-1}.

Thus, we assume that the automaton b has the form presented in Fig. 17, where $A_2\cap B_1\neq\varnothing$.

Let us show that b has an unbounded order. Suppose the contrary. Let $x\in A_2\cap B_1$. Let N be the least natural number such that $b^N=1$ and $w=xx\dots x\dots$. Consider the words

$$w,\ w^b,\ w^{b^2},\dots,\ w^{b^{N-1}},\ w^{b^N}=w.$$

Note that

$$w^{b^l}=y_1y_2\dots y_lx\dots xx\dots,$$

where $y_1,\dots,y_l$ are certain symbols; i.e., no power of the automaton b changes the tail of the sequence w. More precisely, let

$$w^{b^l}=y_1y_2\dots y_{d(l)}x\dots x,$$

where $y_{d(l)}\neq x$. The number $d(l)$ is called the defect of the sequence w^{b^l}. Among the sequences $w,w^b,\dots,w^{b^l},\dots,w^{b^N}$, there exists one with the maximum defect. Let this be w^{b^s}; then the sequence $w^{b^{s+1}}$, $s+1<N$, has lesser defect.

In addition, let the sequences $w^{b^t},w^{b^{t+1}},\dots,w^{b^{s-1}}$, $t\le s$, have the same (maximum) defect, while the sequence $w^{b^{t-1}}$ has a lesser defect.

Thus,

$$\begin{array}{rl|l}
w^{b^{t-1}} &= y_1^{(t-1)}\dots y_{d(t-1)}^{(t-1)}\dots x & x\dots x\dots,\\
w^{b^t} &= y_1^{(t)}\dots y_{d(t)-1}^{(t)}y_{d(t)}^{(t)} & x\dots x\dots,\\
 & \dots & \\
w^{b^s} &= y_1^{(s)}\dots y_{d(s)-1}^{(s)}y_{d(s)}^{(s)} & x\dots x\dots,\\
w^{b^{s+1}} &= y_1^{(s+1)}\dots y_{d(s+1)}^{(s+1)}\dots x & x\dots x\dots.
\end{array}$$

Since σ is a cyclic permutation of the symbols of the input alphabet, there are all symbols of the alphabet X except the letter x among the symbols $y_{d(t)}^{(t)},\dots,y_{d(s)}^{(s)}$; in particular, there are two neighboring symbols $y_{d(q)}^{(q)}$ and $y_{d(q+1)}^{(q+1)}$ such that $y_{d(q)}^{(q)}\neq y_{d(q+1)}^{(q+1)}$ and $y_{d(q)}^{(q)}\in A_1$. This implies that the automaton was in the state b when reading the letter $y_{d(q)}^{(q)}$; therefore, in the next moment, it will remain in the same state. Thus,

$$w^{b^{q+1}}=y_1^{(q+1)}\dots y_{d(q+1)}^{(q+1)}\cdot\Big[(x\dots x\dots)^b\Big]=y_1^{(q+1)}\dots y_{d(q+1)}^{(q+1)}(x^\sigma)\cdot x\dots x\dots,$$

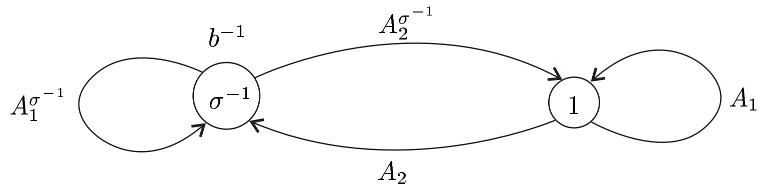


Fig. 19

and we see that the defect of the sequence $w^{b^{q+1}}$ is greater than $d(s)$ in spite of the assumption. This proves that the order of the automaton b is unbounded.

4. Now, consider the last case, when $\varnothing \neq A_1 = B_1$, $A_2 = B_2$, and $A_1 \neq X$. However, in this case, the inverse automaton b^{-1} has the form shown in Fig. 19, and, since $A_1^{\sigma^{-1}} \neq A_1$, this case is reduced to the preceding one. The proposition is proved. $\square$

4.6. Groups generated by linear automata.

Definition 4.4. A *linear automaton* over a field $\Bbbk$ is the set $L = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q, \lambda, \pi\rangle$, where

(i) X_{I}, X_{O}, and Q are vector spaces over $\Bbbk$ (the input and output alphabets and the set of states, respectively);

(ii) $\lambda\colon X_{\mathrm{I}} \oplus Q \to X_{\mathrm{O}}$ is a linear mapping (the output function);

(iii) $\pi\colon X_{\mathrm{I}} \oplus Q \to Q$ is a linear mapping (the transition function).

When X_{I}, X_{O}, and Q are finite-dimensional spaces over a finite field, a linear automaton with a finite number of states can be considered as a finite synchronous automaton.

An example of a finite synchronous automaton that is linear is shown in Fig. 6; this automaton defines the lamplighter group, i.e., the metabelian group $\mathbb{Z} \wr \mathbb{Z}_2$.

This automaton can be defined as an automaton over the field $\mathbb{Z}_2$ with the one-dimensional input and output alphabet $X = \mathbb{Z}_2$ and one-dimensional space of internal states $Q = \mathbb{Z}_2$. The output and transition functions are defined by the equalities

$$\lambda(x,q) = x + q, \qquad \pi(x,q) = x.$$

The states a and b in the figure correspond to 1 and $0 \in \mathbb{Z}_2$.

As in the case of ordinary synchronous automata, an initial linear automaton is the linear automaton with a fixed initial state $q_0 \in Q$. The initial linear automaton also transforms the words over the alphabet X_{I} into the words over the alphabet X_{O}.

In this case, the set of finite words X_{I}^n (X_{O}^n) is interpreted as a direct sum of spaces together with the natural embedding $X_{\mathrm{I}}^{n-1} \hookrightarrow X_{\mathrm{I}}^n$ that identifies the word $x_1x_2\dots x_{n-1}$ with the word $x_1x_2\dots x_{n-1}0$. In particular, an empty word is identified with the null vector. The sets X_{I}^* and X_{O}^* are the inductive limits of the constructed direct spectra $\{X_{\mathrm{I}}^n\}$ and $\{X_{\mathrm{O}}^n\}$, respectively. The spaces of infinite sequences X_{I}^ω and X_{O}^ω are also considered as vector spaces with coordinatewise actions. In this case, X_{I}^* is a subspace of X_{I}^ω.

For an arbitrary linear automaton, the following decompositions hold: $\lambda(x,q) = \lambda(0,q) + \lambda(x,0)$ and $\pi(x,q) = \pi(0,q) + \pi(x,0)$.

Using these formulas and the definition of the action of the automaton on finite and infinite words, we can readily derive the following proposition.

Proposition 4.15 [61]. *The mappings* $\lambda\colon X_{\mathrm{I}}^n \oplus Q \to X_{\mathrm{O}}^n$ *and* $\pi\colon X_{\mathrm{I}}^n \oplus Q \to Q$ *are linear.*

The mapping $\lambda\colon X_{\mathrm{I}}^\omega \oplus Q \to X_{\mathrm{O}}^\omega$ *is also linear and is decomposed into the sum* $\lambda(w,q) = \lambda(0,q) + \lambda(w,0)$.

Hence we obtain the following assertions.

Proposition 4.16. *For an arbitrary linear automaton* A, *the group* $G(A)$ *defined by this automaton is the extension of an abelian group by a cyclic group.*

Proof. Since $\lambda(w,q) = \lambda(w,0) + \lambda(0,q)$, for every state q, the initial automaton A_q acts on the space X^ω as a certain affine transformation $w \mapsto \phi(w)+v$; here, the linear transformation $\phi(w)$ is independent of the state q. Hence, all transformations of the space X^ω belonging to the group $G(A)$ are given by $w \mapsto \phi^n(w) + v$ for certain $n \in \mathbb{Z}$ and $v \in X^\omega$. This completes the proof. □

Proposition 4.17. *Let the alphabet* X *be a one-dimensional vector space. Then, any group generated by linear automata is metabelian.*

Proof. If λ is the output function of a linear automaton, we have the decomposition $\lambda(x,q) = \lambda(x,0) + \lambda(0,q)$. The linear mapping $\lambda(\,\cdot\,,0)\colon X \to X$ is scalar; i.e., for a certain $k \in \Bbbk$, the law $\lambda(x,0) = k \cdot x$ holds because the space X is one-dimensional. This implies that an arbitrary initial linear automaton acts on X^ω by the affine transformation of the form $w \mapsto k \cdot w + v$ for a certain $v \in X^\omega$. However, the group of transformations of this type is metabelian. □

It is convenient to consider the elements of X_{I}^* and X_{O}^* as polynomials over X_{I} and X_{O}, respectively, by identifying the word $x_0x_2\dots x_n$ with the expression $x_0 + x_1t + x_2t^2 + \dots + x_nt^n$. This identification is consistent with the embedding $X_{\mathrm{I}}^{n-1} \hookrightarrow X_{\mathrm{I}}^n$ described above. Denote by $X_{\mathrm{I}}[t]$ and $X_{\mathrm{O}}[t]$ the sets of polynomials over X_{I} and X_{O}, respectively. In addition to the structure of the vector space over the field $\Bbbk$, the natural structure of the $\Bbbk[t]$-module is also defined on these sets.

Similarly, the spaces X_{I}^ω and X_{O}^ω are identified with the spaces of formal power series $X_{\mathrm{I}}[[t]]$ and $X_{\mathrm{O}}[[t]]$, respectively, which are $\Bbbk[[t]]$-modules.

Using this interpretation, one can obtain a more accurate characterization of the mappings defined by linear automata.

Proposition 4.18 [18]. *Let* $L = \langle X_{\mathrm{I}}, X_{\mathrm{O}}, Q, \lambda, \pi\rangle$ *be a linear automaton over the field* $\Bbbk$. *Then, the mapping* $\lambda(\cdot,0)\colon X_{\mathrm{I}}[[t]] \to X_{\mathrm{O}}[[t]]$ *defined by this automaton is a morphism of* $\Bbbk[[t]]$*-modules.*

Proof. By Proposition 4.15, the mapping $\lambda(\,\cdot\,,0)$ is $\Bbbk$-linear; therefore, it suffices to show that it commutes with the multiplication by t.

Let $w = x_0+x_1t+x_2t^2+\dots \in X_{\mathrm{I}}[[t]]$. We obtain a sequence of internal states $\{q_0 = 0, q_1, q_2, \dots\}$ defined by the equalities $q_{n+1} = \pi(x_n, q_n) = \pi(0,q_n) + \pi(x_n,0)$. Then, the image of w under the action of L is equal to $y_0 + y_1t + y_2t^2 + \dots$, where $y_n = \lambda(x_n,q_n) = \lambda(0,q_n) + \lambda(x_n,0)$.

These formulas imply that, for $tw = 0 + x_0t + x_1t^2 + \dots$, the corresponding sequence of states is equal to $\{0,0,q_1,q_2,\dots\}$; therefore, the image of the word tw is equal to $0 + y_0t + y_1t^2 + \dots = t(y_0 + y_1t + y_2t^2 + \dots)$. Thus, $\lambda(tw,0) = t\lambda(w,0)$, as required. □

As an example, consider the case of the automaton that defines the lamplighter group. Let $w = x_0 + x_1t + x_2t^2 + \dots \in \mathbb{Z}_2[[t]]$. Since $\lambda(x,q) = x + q$ and $\pi(x,q) = x$, the sequence of states of the automaton is equal to $\{0, x_0, x_1, x_2, \dots\}$, while $\lambda(w,0) = y_0 + y_1t + y_2t^2 + \dots$, where $y_0 = x_0$ and $y_n = x_{n-1} + x_n$ for $n \geq 1$; hence, $\lambda(w,0) = (1+t)w$.

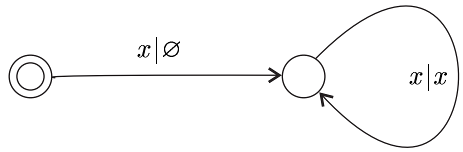


Fig. 20. Unilateral shift

5. EXAMPLES OF ASYNCHRONOUS AUTOMATON GROUPS

5.1. Groups of shift automorphisms. A *bilateral sequence* over the alphabet X is the sequence of the form $\ldots x_{-2}x_{-1}\cdot x_0x_1x_2\ldots$, where the dot indicates the position between the zeroth and the (-1)th coordinates. The set of all bilateral sequences over the alphabet X is denoted by $X^{\mathbb{Z}}$ and is equipped with the Tikhonov topology.

A *bilateral Bernoulli shift* is the transformation σ defined by the relation

$$(\ldots x_{-2}x_{-1}\cdot x_0x_1x_2\ldots)^{\sigma}=(\ldots x_{-2}x_{-1}x_0\cdot x_1x_2\ldots).$$

Obviously, a bilateral shift is a homeomorphism of the space $X^{\mathbb{Z}}$.

A *unilateral shift* is the transformation of the space of unilateral infinite sequences X^{ω} defined by the rule

$$(x_1x_2\ldots)^{\sigma}=x_2x_3\ldots.$$

It is clear that a unilateral shift is defined by a finite asynchronous automaton with the diagram shown in Fig. 20.

The initial state of this automaton, irrespective of the incoming letter, outputs an empty word; after that, the automaton outputs the incoming words without changes.

Let $\nu_0\colon X^{\mathbb{Z}}\to X^{\omega}$ be a mapping defined by the equality

$$(\ldots x_{-2}x_{-1}\cdot x_0x_1x_2\ldots)^{\nu_0}=x_0x_{-1}x_1x_{-2}x_2x_{-3}\ldots, \tag{7}$$

where the odd positions on the right-hand side are occupied by all nonnegative coordinates in the increasing order of their numbers, whereas the even positions are occupied by all negative coordinates in the decreasing order of their numbers. The mapping ν_0 is a homeomorphism.

Similarly we define the mapping $\nu_1\colon X^{\mathbb{Z}}\to X^{\omega}$:

$$(\ldots x_{-2}x_{-1}\cdot x_0x_1x_2\ldots)^{\nu_1}=x_{-1}x_0x_{-2}x_1x_{-3}x_2\ldots, \tag{8}$$

which also is a homeomorphism.

A homeomorphism α of the space $X^{\mathbb{Z}}$ is called a *shift endomorphism* if it commutes with the shift.

A mapping is a shift endomorphism if and only if it is defined by a certain *cellular automaton*, i.e., if there exist numbers $m,n\in\mathbb{N}$ (called *anticipation* and *memory*, respectively) and a function $F\colon X^{n+m+1}\to X$ such that if

$$(\ldots x_{-2}x_{-1}\cdot x_0x_1x_2\ldots)^{\alpha}=(\ldots y_{-2}y_{-1}\cdot y_0y_1y_2\ldots),$$

then $y_i=F(x_{i-n},x_{i-n+1},\ldots,x_{i+m})$.

Let $\alpha\colon X^{\mathbb{Z}}\to X^{\mathbb{Z}}$ be a shift endomorphism. For every pair of finite words $v_-=a_{-r}a_{-r+1}\ldots a_{-1}$, $v_+=a_0a_1\ldots a_s\in X^*$, define the mapping $\alpha|_{(v_-,v_+)}\colon X^{\mathbb{Z}}\to X^{\mathbb{Z}}$ such that the relation

$$(\ldots x_{-2}x_{-1}\cdot x_0x_1x_2\ldots)^{\alpha|_{(v_-,v_+)}}=(\ldots y_{-2}y_{-1}\cdot y_0y_1y_2\ldots)$$

holds if and only if

$$(\ldots x_{-2}x_{-1}(a_{-r}a_{-r+1}\ldots a_{-1}\cdot a_0a_1\ldots a_s)x_0x_1x_2\ldots)^\alpha$$
$$=(\ldots y_{-2}y_{-1}(b_{-r}b_{-r+1}\ldots b_{-1}\cdot b_0b_1\ldots b_s)y_0y_1y_2\ldots),$$

where $b_{-r}b_{-r+1}\ldots b_{-1}b_0b_1\ldots b_s$ is a certain word.

Lemma 5.1. *For any shift endomorphism $\alpha\colon X^{\mathbb{Z}}\to X^{\mathbb{Z}}$, the set of all possible mappings of the form $\alpha|_{(v_-,v_+)}$ is finite.*

Proof. If $m,n\in\mathbb{N}$ are, respectively, the anticipation and the memory of a cellular automaton that defines the endomorphism α, then $\alpha|_{(v_-,v_+)}$ depends only on the first m letters of the word v_- and the last n letters of the word v_+; therefore, the number of different mappings of the form $\alpha|_{(v_-,v_+)}$ is finite. □

Lemma 5.1 and Theorem 2.5 imply the following proposition.

Proposition 5.2. *Let ν_0 be the homeomorphism defined by equality* (7) *and α be a bilateral shift endomorphism. Then, the continuous mapping $\nu_0^{-1}\alpha\nu_0\colon X^\omega\to X^\omega$ is rational.*

Proof. Let $v=a_1a_2\ldots a_k\in X^*$ be an arbitrary finite word. If k is even, then the restriction $(\nu_0^{-1}\alpha\nu_0)|_v$ is equal to $\nu_0^{-1}\alpha|_{(v_-,v_+)}\nu_0$, where

$$v_-=a_ka_{k-2}\ldots a_4a_2,$$
$$v_+=a_1a_3\ldots a_{k-3}a_{k-1}.$$

If k is odd, then the restriction $(\nu_0^{-1}\alpha\nu_0)|_v$ is equal to $\nu_1^{-1}\alpha|_{(v_-,v_+)}\nu_1$, where

$$v_-=a_{k-1}a_{k-3}\ldots a_4a_2,$$
$$v_+=a_1a_3\ldots a_{k-2}a_k,$$

and ν_1 is the mapping defined by (8).

Thus, by Lemma 5.1, there is a finite number of restrictions $(\nu_0^{-1}\alpha\nu_0)|_v$; hence, by Theorem 2.5, $\nu_0^{-1}\alpha\nu_0$ is rational. □

The following proposition is a particular case of Proposition 5.2.

Proposition 5.3. *Let σ be a bilateral shift. Then, the homeomorphism $\nu_0^{-1}\sigma\nu_0\colon X^\omega\to X^\omega$ is rational.*

Proposition 5.2 also implies the following theorem.

Theorem 5.4. *The semigroup of bilateral Bernoulli shift endomorphisms is isomorphic to a subsemigroup of the semigroup of finite asynchronous automata $\mathcal{F}$.*

The concept of the unilateral shift endomorphism is defined in a similar way. A mapping $\alpha\colon X^\omega\to X^\omega$ is the unilateral shift endomorphism if and only if it is defined by a cellular automaton with zero memory.

Note that the set of all bilateral shift endomorphisms with zero memory is a semigroup. This semigroup is isomorphic to the semigroup of unilateral shift endomorphisms. Similarly, the set of bilateral shift endomorphisms with zero anticipation is a semigroup isomorphic to the semigroup of endomorphisms with zero memory.

The endomorphism of (unilateral or bilateral) shift is called its *automorphism* if it is a homeomorphism. The set of all shift automorphisms is a group. The group of unilateral shift automorphisms is a group of invertible elements of its semigroup of endomorphisms and, hence, is isomorphic to the group of invertible elements of the semigroup of bilateral shift endomorphisms with zero memory.

The groups of unilateral and bilateral shift automorphisms possess the following properties (see [35]).

- The group of unilateral shift automorphisms has the order 2 for $|X| = 2$. If $|X| > 2$, then this group is infinite, countable, and residually finite.
- The group of bilateral shift automorphisms contains the isomorphic copy of any finite group.
- A free product of a finite number of cyclic groups is embeddable into the group of unilateral or bilateral shift automorphisms over the alphabet of sufficiently large cardinality. In particular, a free product of three cyclic groups of order 2 is embeddable into the group of unilateral or bilateral shift automorphisms for $|X| \geq 6$.
- A finite group G is embeddable into the group of unilateral shift automorphisms over the alphabet X if and only if all composition quotients of G are isomorphic to the subgroups of the symmetric group $\mathrm{Sym}(X)$.
- The automorphism group of an arbitrary full bilateral shift is embeddable into the group of any other full bilateral shift automorphisms.

Note that the question whether the groups of full bilateral shift automorphisms are isomorphic for different alphabets of cardinality ≥ 2 remains open.

Theorem 5.4 entails the following fact.

Corollary 5.5. *The group of bilateral shift automorphisms is isomorphic to a certain subgroup of the group of rational automorphisms $\mathcal{Q}$.*

In the case of a unilateral shift, we obtain a stronger statement.

Theorem 5.6 [48]**.** *The semigroup of full unilateral shift endomorphisms over the alphabet X is isomorphic to the subsemigroup of the semigroup of finite synchronous automata $\mathcal{SF}(X)$.*

Proof. Let P_n be the set of all infinite sequences $x_1x_2\ldots \in X^\omega$ such that $x_i = x_n$ for any $i > n$.

For any $n \in \mathbb{N}$, construct the bijection $\rho_n\colon X^n \to P_n$:

$$(x_1x_2\ldots x_n)^{\rho_n} = x_nx_{n-1}\ldots x_2x_1x_1x_1\ldots .$$

Let α be an arbitrary unilateral shift endomorphism on X^ω. It follows from the definition of a cellular automaton with zero memory that the set P_n is invariant with respect to α; i.e., $P_n^\alpha \subseteq P_n$. Hence, α induces on X^n the mapping $\alpha_n = \rho_n^{-1}\alpha\rho_n$. It is obvious that the mappings α_n preserve the common beginnings and the lengths of words; therefore, it follows from Proposition 3.1 that α_n in aggregate define a synchronous automatic transformation of the set X^* and, hence, induce a synchronous automatic transformation of the set X^ω. Denote the latter transformation by α^*. It directly follows from the definitions that the mapping $\alpha \mapsto \alpha^*$ is a homomorphism of semigroups.

Let $F\colon X^{m+1} \to X$ be a function that defines the endomorphism α (m is a anticipation); in this case, if $(x_1x_2\ldots)^{\alpha^*} = y_1y_2\ldots$, then

$$y_i = F(x_i, x_{i-1}, \ldots, x_{i-m}),$$

where $x_j = x_1$ for $j \leq 0$.

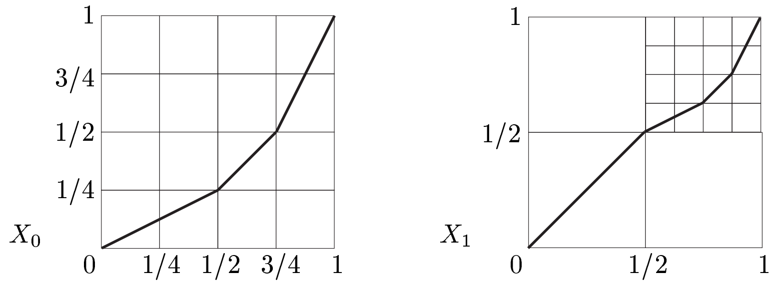


Fig. 21. Generators of the Thompson group

Let the set of states of the automaton A_{q_0} be equal to the set X^m with the added initial state $q_0 \notin X^m$. Let us define the transition and output functions by the equalities

$$\begin{aligned}\pi(x, q_0) &= xx\ldots x,\\ \pi(x, x_1x_2\ldots x_n) &= x_2x_3\ldots x_nx,\\ \lambda(x, q_0) &= F(x, x, \ldots x),\\ \lambda(x, x_1x_2\ldots x_n) &= F(x, x_n, x_{n-1}, \ldots, x_1).\end{aligned}$$

It is clear that, under this definition, the action of the automaton A_{q_0} coincides with the action of the transformation α^*. $\square$

Now, Theorem 5.6 implies the following assertion.

Corollary 5.7. *The automorphism group of a full unilateral shift over the alphabet X is isomorphic to a certain subgroup of the group of finite synchronous automata $\mathcal{FGA}(X)$.*

5.2. The Thompson groups. In 1965, when dealing with the problems of logic, R. Thompson discovered remarkable groups that have been called the Thompson groups. Later on, these groups and their generalizations were studied by G. Higman, M. Brin, C. Squier, K. Brown, R. Geoghegan, E. Ghys, V. Sergiescu, J. Cannon, W. Floyd, W. Parry, V. Guba, M. Sapir and many other mathematicians. These groups have found applications in topology, cohomology theory, theory of dynamical systems, analysis, and other fields of mathematics (see [14]).

The *Thompson group F* is a group with the composition operation that consists of increasing piecewise linear homeomorphisms f of the segment $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$, with a finite number of nondifferentiability points that are dyadically rational (i.e., have the form $\frac{m}{2^n}$, $m, n \in \mathbb{N}$), the derivative, at all points where it exists, being an integer power of two.

The group F is generated by two functions X_0 and X_1 with the graphs illustrated in Fig. 21; the graph of the function X_1 on the interval $[1/2, 1]$ is similar to the graph of the function X_0.

Define an infinite sequence X_n, $n \geq 0$, of elements of the group F by the condition that the graph of the function X_n on the interval $[0, 1-2^{-n}]$ is directed along the diagonal, whereas, on the interval $[1-2^{-n}, 1]$, it is similar to the graph of the function X_0. Then, F has the following presentation by generating elements and defining relations:

$$F = \langle X_0, X_1, \ldots :\ X_jX_iX_j^{-1} = X_{i+1},\ j < i\rangle.$$

In fact, F is a finitely presented group and can be described by the balanced presentation with two generating elements and two relations

$$F = \langle a, b :\ [ab^{-1}, a^{-1}ba] = [ab^{-1}, a^{-2}ba^2] = 1\rangle.$$

The group F has many interesting properties. It does not contain free subgroups of rank 2, no nontrivial law holds in this group, and its arbitrary subgroup is either free abelian or contains the wreath product $\mathbb{Z} \wr \mathbb{Z}$ and, hence, a free abelian subgroup of arbitrary finite rank. All nontrivial normal subgroups of F contain its commutator subgroup F', which is a simple group and such that $F/F' \simeq \mathbb{Z}^2$. The question whether the group F is amenable remains open.

The Thompson group can also be considered as a group of transformations of the boundary of a binary tree. Let us identify infinite sequences over the alphabet $X = \{0,1\}$ with the numbers from the interval $[0,1]$ in binary notation by the mapping

$$\phi\colon x_1x_2x_3\ldots \mapsto \sum_{k=1}^{\infty} x_k \cdot 2^{-k}.$$

Let $D = [0,1] \cap \mathbb{Z}[\frac{1}{2}]$ be the set of all dyadically rational numbers from the interval $[0,1]$. If $a \notin D$, then there exists a unique sequence $w \in X^\omega$ such that $\phi(w) = a$. Otherwise, a has two preimages: one of the form $x_1x_2\ldots x_n1000\ldots$ and the other of the form $x_1x_2\ldots x_n0111\ldots$.

For $w \notin \phi^{-1}(D)$ and $g \in F$, define $w^g = \phi^{-1}(\phi(w)^g)$. Extending, by continuity, the obtained action of the group F onto the entire space X^ω, we obtain a well-defined action of F by homeomorphisms of the Cantor set X^ω.

It is easily seen that the homeomorphism corresponding to the generator X_0 is defined on infinite words by the relations

$$\begin{aligned}(0w)^{X_0} &= 00w,\\ (10w)^{X_0} &= 01w,\\ (11w)^{X_0} &= 1w,\end{aligned}$$

where $w \in X^\omega$ is an arbitrary infinite word.

Similarly, the generator X_1 is defined by the relations

$$\begin{aligned}(0w)^{X_1} &= 0w,\\ (10w)^{X_1} &= 100w,\\ (110w)^{X_1} &= 101w,\\ (111w)^{X_1} &= 11w.\end{aligned}$$

Moreover, for an arbitrary element Y of the group F, there exist two sets of finite words $(v_1, v_2, \ldots, v_n)$ and $(u_1, u_2, \ldots, u_n)$ such that, for any infinite word w, exactly one of the words v_i and exactly one of the words u_i are the prefixes of w; in addition, the action of Y is defined by the relations

$$(v_iw')^Y = u_iw', \tag{9}$$

where $w = v_iw'$. The matrix $\left(\begin{smallmatrix} v_1 & v_2 & \cdots & v_n \\ u_1 & u_2 & \cdots & u_n\end{smallmatrix}\right)$ is called the *tableau* of an element of the Thompson group.

It is geometrically convenient to represent the action of the elements of the Thompson group F as a permutation of the subtrees of a binary tree. For example, the generator X_0 acts on a tree as is shown in Fig. 22 (the numbers indicate where the appropriate subtrees are translated).

The matrix $\left(\begin{smallmatrix} v_1 & v_2 & \cdots & v_n \\ u_1 & u_2 & \cdots & u_n\end{smallmatrix}\right)$ represents the tableau of a certain element of the group F if the corteges $(v_1, v_2, \ldots, v_n)$ and $(u_1, u_2, \ldots, u_n)$ define a partition of the space X^ω (i.e., if, for every word $w \in X^\omega$, there exist exactly one element u_i of the first cortege and exactly one element v_j of the second cortege that are the prefixes of w) and if both corteges are ascending ones in lexicographic

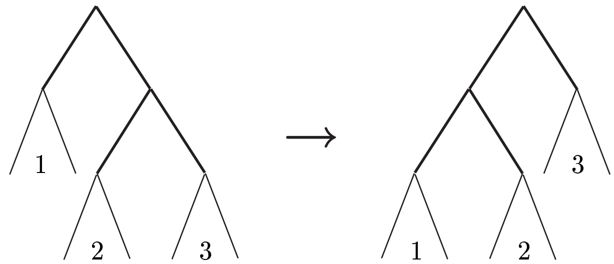


Fig. 22. The action of X_0 on a tree

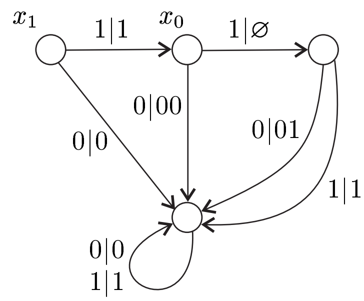


Fig. 23. Automata that generate the Thompson group F

order, where 0 is assumed to be less than 1. Note that the word v_i is less than the word v_j if and only if $\phi(v_i000\ldots) < \phi(v_j000\ldots)$.

In addition, the set of transformations defined by a certain tableau $\left(\begin{smallmatrix} v_1 & v_2 & \cdots & v_n \\ u_1 & u_2 & \cdots & u_n \end{smallmatrix}\right)$, not necessarily satisfying the monotonicity condition, also is a group. This group is called the Thompson group V (or the partition group) and contains the group F as a subgroup. This group is also finitely presented and, in addition, is simple. In [62], the group V and its generalizations were applied to the analysis of groups with solvable word problems.

The group V is generated by the elements of F and the transformations defined by the tableaux

$$\begin{pmatrix} 0 & 10 & 11 \\ 10 & 0 & 11 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 10 & 110 & 111 \\ 0 & 110 & 10 & 111 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 10 & 11 \\ 0 & 11 & 10 \end{pmatrix}.$$

In addition to the groups F and V, we also consider an intermediate group T, $F < T < V$, that consists of elements defined by the tableaux $\left(\begin{smallmatrix} v_1 & v_2 & \cdots & v_n \\ u_1 & u_2 & \cdots & u_n \end{smallmatrix}\right)$ in which the sequences $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ become ascending in lexicographic order after certain cyclic permutations. The group T is the first example of a finitely presented simple group.

The homeomorphism of the Cantor set that is defined by an arbitrary element of the Thompson group V (and, hence, by elements of the groups F and T) is rational. Figure 23 represents the automata that define the transformations generating the group F. The automata with the initial states x_0 and x_1 define the generators X_0 and X_1, respectively.

Thus, the following proposition holds.

Proposition 5.8. *In the group $\mathcal{Q}$ of rational automorphisms of the Cantor set, there exist subgroups isomorphic to the Thompson groups V, T, and F. In particular, the group $\mathcal{Q}$ is not residually finite.*

Using Theorem 1.10 from [62], one can prove the following theorem.

Theorem 5.9 [56]**.** *A subgroup in $\mathcal{Q}$ generated by the Thompson group V and the Grigorchuk group is a finitely presented simple group.*

Recently, C. Röver has demonstrated that this group is isomorphic to an abstract commensurator of the Grigorchuk 2-group.

6. ACTIONS ON ROOTED TREES

6.1. Groups acting on rooted trees. Groups and semigroups of automata act naturally on regular rooted trees. One of approaches to the investigation of these actions, based on the application of the length function, was proposed in [54]. Here, we apply a more geometrical approach and consider a wider class of trees, focusing on the actions on the boundary of a tree.

Let T be a locally finite rooted tree, i.e., a tree with a fixed vertex, denoted henceforth by v_0. On the set of vertices, a combinatorial distance is defined that is equal to the number of edges in the shortest path connecting two vertices. The set of vertices of a rooted tree is naturally partitioned into *levels* (or *spheres*), where the nth level (the sphere of radius n) is defined as the set L_n of vertices that are situated at the distance n from the root. In particular, the zero level consists of the root vertex. The level of a vertex v is denoted by $|v|$.

A vertex v lies *below* vertex u if the path connecting the vertex v with the root passes through u. A subtree that consists of vertices that lie below the vertex v (which serves as the root) with the same edges as in T is denoted by T_v and is called a *subtree with the root vertex* v (see Fig. 24).

The *automorphism* of the rooted tree is defined as the bijection of the set of vertices that preserves the position of the root and the incidence relation. The group of all automorphisms is denoted by $\operatorname{Aut} T$. It directly follows from the definitions that the levels are invariant with respect to the automorphisms of the rooted tree.

A group G acting by automorphisms on a rooted tree is called *spherically transitive* if it is transitive on spheres.

A tree T is called *spherically homogeneous* (or *spherically transitive*) if its full group of automorphisms is spherically transitive.

If a tree T is spherically homogeneous, then the valences of all vertices of the same level are equal. The *spherical index* (or *branch index*) of a spherically homogeneous tree T is defined as a sequence of natural numbers $\overline{m} = (m_1, m_2, \ldots)$, where m_1 is the valence of the root, while, for $n > 1$, $m_n + 1$ is the valence of every vertex of the level $(n-1)$. In other words, m_n defines the branch index of each vertex of the level $(n-1)$.

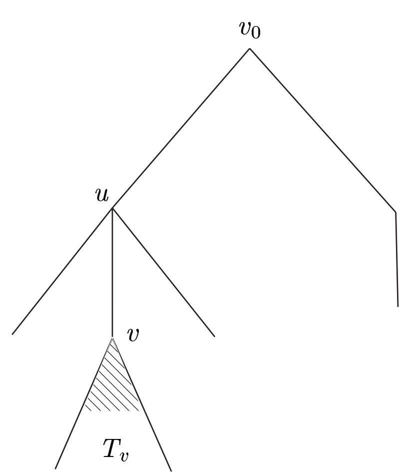


Fig. 24. A rooted tree

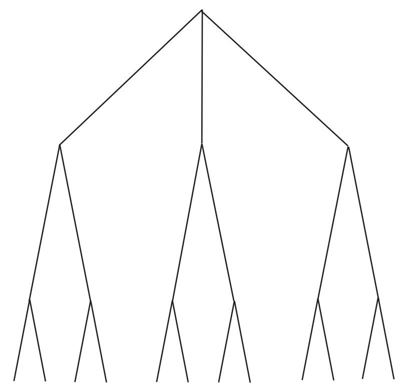


Fig. 25. A spherically homogeneous tree

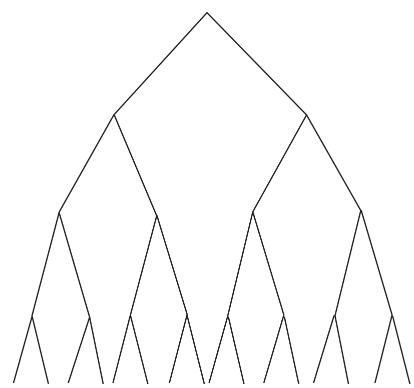


Fig. 26. A binary tree

We will consider only the trees such that $m_n > 1$.

In the case of a finite spherically homogeneous rooted tree, the spherical index is a finite sequence.

Any sequence $\overline{m} = \{m_n\}_{n=1}^{\infty}$ of natural numbers defines a rooted spherically homogeneous tree.

Figure 25 shows an example of a (finite) rooted spherically homogeneous tree with the spherical index $(3, 2, 2)$.

A spherically homogeneous rooted tree is called *regular* if all components m_i of its spherical index are equal. An example of the regular tree is given by a *binary* tree (a tree with the spherical index $(2, 2, \ldots)$, see Fig. 26).

Let $\mathbf{X} = \{X_n\}_{n=1}^{\infty}$ be a sequence of sets such that $|X_n| = m_n$. Denote $\mathbf{X}^0 = \{\varnothing\}$, $\mathbf{X}^n = X_1 \times X_2 \times \ldots \times X_n$, and $\mathbf{X}^* = \bigcup_{n=0}^{\infty} \mathbf{X}^n$. We identify the elements of the set $\mathbf{X}^n$ with the words of the form $x_1 x_2 \ldots x_n$, where $x_i \in X_i$. The corresponding rooted tree is denoted by $T(\mathbf{X})$. A rooted tree with the spherical index $\overline{m}$ is isomorphic to the tree $T(\mathbf{X})$ for an appropriate sequence $\mathbf{X}$.

If the sequence $\mathbf{X} = (X, X, \ldots)$ is constant, then the vertices of the regular tree $T(\mathbf{X})$ are the words over the alphabet X. Recall that, by Proposition 3.6, synchronous automata over the alphabet X define in this case the automorphisms of the tree $T(\mathbf{X})$ by their action on X^*, while the automorphism group $\operatorname{Aut} T(\mathbf{X})$ is isomorphic to the group of synchronous automata $\mathcal{GA}(X)$. Similarly, for any noninitial synchronous automaton A, the group $G(A)$ defined by the automaton A acts by automorphisms on the tree $T(\mathbf{X})$.

The action of the automorphism $g \in \operatorname{Aut} T$ on the vertex v is denoted by v^g.

Definition 6.1. Suppose that the group G acts by automorphisms on a rooted tree T and u is a vertex of the tree.

- A *stabilizer of the vertex* u is the subgroup $\mathrm{St}_G(u) = \{g \in G\colon\ u^g = u\}$.
- A *stabilizer* $\mathrm{St}_G(n)$ *of the nth level* is the intersection of all vertex stabilizers of this level.
- A *rigid stabilizer of the vertex* u is the group $\mathrm{rist}_G(u)$ of automorphisms from G that act trivially on the complement to the subtree T_u.
- A *rigid stabilizer of the nth level*, $\mathrm{rist}_G(n)$, is a subgroup generated by all rigid stabilizers of the vertices of this level.

Henceforth, we will omit index G when this does not lead to confusion.

The stabilizers and rigid stabilizers have the following elementary properties.

Proposition 6.1. *Let $G \le \mathrm{Aut}\, T$. The level stabilizer $\mathrm{St}_G(n)$ is a normal subgroup of finite index in G. The intersection $\bigcap_{n=0}^{\infty} \mathrm{St}_G(n)$ is a trivial group.*

A rigid level stabilizer $\mathrm{rist}_G(n)$ is a normal subgroup of G and is equal to a direct product of the rigid vertex stabilizers of this level.

Definition 6.2. A spherically transitive group $G \le \mathrm{Aut}\, T$ is called a *branch* group if $\mathrm{rist}(n)$ is a subgroup of finite index for every $n \in \mathbb{N}$.

A spherically transitive group $G \le \mathrm{Aut}\, T$ is called *weakly branch* if $|\mathrm{rist}(n)| = \infty$ for every $n \in \mathbb{N}$.

Examples of branch groups are given by the entire group $\mathrm{Aut}\, T$ of automorphisms of a spherically transitive tree T, the group of finite synchronous automata $\mathcal{FGA}(X)$, the Grigorchuk group, the groups defined by the automata depicted in Figs. 9 and 15, the Gupta–Sidki group, and others.

The branch groups constitute an important class of residually finite groups that includes the examples of just infinite groups (the groups in which all proper quotients are finite), the Burnside-type groups, the groups of intermediate growth, groups of finite width, and the groups possessing other interesting properties (see [24]).

6.2. Boundary of a tree. An *end* of a rooted tree T is the infinite path without repetitions that starts at the root vertex. The set of ends is called the *boundary of a tree* and is denoted by ∂T. For a tree of the form $T(\mathbf{X})$, the ends are naturally identified with infinite sequences of the form $x_1 x_2 \dots$, where $x_i \in X_i$; i.e., the boundary $\partial(T(\mathbf{X}))$ is identified with the Cartesian product $\mathbf{X}^\omega = \prod_{n=0}^{\infty} X_n$.

If the tree T is spherically transitive, then $\mathrm{Aut}\, T$ acts transitively on the boundary ∂T.

The *end stabilizer* $\gamma \in \partial T$ in the group $G \le \mathrm{Aut}\, T$ is called a *parabolic subgroup* of the group G and is denoted by $\mathrm{St}_G(e)$, or P, if it is clear what group and what end are concerned.

Let $\overline{\lambda} = \{\lambda_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of positive real numbers that tends to zero. Define the metric $d_{\overline{\lambda}}$ on the boundary: $d_{\overline{\lambda}}(\gamma, \gamma) = 0$ and $d_{\overline{\lambda}}(\gamma_1, \gamma_2) = \lambda_n$, where n is the number of the level at which the paths $\gamma_1, \gamma_2 \in \partial T$ diverge.

The space $(\partial T, d_{\overline{\lambda}})$ is a compact ultrametric space. Every open ball of radius λ_n of this space (i.e., a set of the form $\{\gamma \in \partial T\colon\ d(\gamma, \gamma_0) < \lambda_n\}$, where $\gamma_0 \in \partial T$) is naturally identified with the boundary of the subtree T_v for a certain vertex v of the level $n-1$.

The group $\mathrm{Aut}\, T$ acts on ∂T by isometries. In addition, the equality $\mathrm{Aut}\, T = \mathrm{Isom}(\partial T, d_{\overline{\lambda}})$ holds.

The following assertion is known as a part of folklore.

Proposition 6.2. *Let $(\mathcal{X}, d)$ be a compact ultrametric space. Then, there exist a rooted tree T and a (finite if the space is finite and infinite, tending to zero, if the space is infinite) monotonically decreasing sequence of positive numbers $\overline{\lambda} = \{\lambda_n\}$ such that the space $(\mathcal{X}, d)$ is isometric to the space $(\partial T, d_{\overline{\lambda}})$.*

If the isometry group of the space $(\mathcal{X}, d)$ is transitive on $\mathcal{X}$, then the tree T is spherically homogeneous.

Proof. The fact that d is the ultrametric implies that the relation $d(x, y) < R$ on the set $\mathcal{X}$ is an equivalence relation for any positive R, and the open balls of radius R are the equivalence classes of this relation. Therefore, every point of a ball is its center, two balls of the same radius either coincide or do not intersect, and two open balls of the space $(\mathcal{X}, d)$ either do not intersect or one is a subset of the other.

By the hypothesis, the space $\mathcal{X}$ is compact; hence, for every $R > 0$, there exists a finite covering of this space by open balls of radius R. Thus, the set of different open balls of radius R is finite, and all of them are closed and compact.

Let $n(R)$ be the number of different open balls of radius R. For $R > 0$, the function $n(R)$ takes only natural values, is left continuous, and decreases. If the space is infinite, this function tends to infinity as $R \to 0$. Hence, it is piecewise constant, and its set of discontinuity points can be arranged in the decreasing sequence $\lambda_1 > \lambda_2 > \ldots > 0$, which is finite for finite $\mathcal{X}$ and infinite and tends to zero for an infinite space. If $R_1, R_2 \in (\lambda_{i+1}, \lambda_i]$, then the number of balls of radius R_1 is equal to the number of balls of radius R_2; hence, the set of balls of radius R_1 coincides with the set of balls of radius R_2. If $R \in (\lambda_1, +\infty)$, then any ball of radius R coincides with the whole space $\mathcal{X}$.

Let us construct a rooted tree T by identifying its root vertex with $\mathcal{X}$ and the nth level with the set of balls of radius λ_n. Connect two balls from adjacent levels by an edge if and only if one is a subset of the other. Since two balls either do not intersect or one is contained in the other, the graph obtained is a rooted tree. It follows from the aforesaid that any ball of the space $(\mathcal{X}, d)$ coincides with the ball corresponding to one of the vertices of the tree T.

We identify each end $B_0 \supset B_1 \supset B_2 \supset \ldots$ (where B_i are the vertices of the constructed tree, i.e., the balls of the space $(\mathcal{X}, d)$) of the tree T with a point $\bigcap_{n=0}^{\infty} B_n$ of the space $(\mathcal{X}, d)$. It follows from the construction that this identification is a bijection between the boundary of T and the space $(\mathcal{X}, d)$.

If $d(x, y) = \lambda$, then, for any $R > \lambda$, the points x and y lie in the same ball of radius R but belong to different balls of radius λ. Therefore, $\lambda = \lambda_n$ for a certain $n \geq 1$, and the paths corresponding to x and y diverge for the first time at the level with the number n. Thus, the constructed bijection between $\mathcal{X}$ and ∂T is the isometry for the boundary metric defined by the sequence $\overline{\lambda} = \{\lambda_1, \lambda_2, \ldots\}$.

If the group of isometries of the space $(\mathcal{X}, d)$ is transitive, then it is also transitive on the set of balls of fixed radius; therefore, the tree T obtained is spherically homogeneous. □

The construction from this proof can be generalized to the case of a totally disconnected compact *metric* space in the following manner.

Let $\mathcal{C}$ be an arbitrary compact totally disconnected metric space. A finite set of pairwise disjoint clopen sets $\mathsf{A} = \{A_i\}_{i=1}^{n}$ such that $\mathcal{C} = \bigcup_{i=1}^{n} A_i$ is called the *clopen partition* of the space $\mathcal{C}$. The *diameter* of the partition A is the maximum diameter of the sets A_i.

The clopen partition $\{A_i\}_{i=1}^{n}$ is the *refinement* of the partition $\{B_i\}_{i=1}^{m}$ if every set B_i is a union of certain sets A_i.

The *clopen covering* of the space $\mathcal{C}$ is a finite set of clopen sets $\{U_i\}_{i=1}^{n}$ such that $\mathcal{C} = \bigcup_{i=1}^{n} U_i$.

Let $\{U_i\}_{i=1}^n$ be a clopen covering of the space $\mathcal{C}$. The clopen partition $\{A_j\}_{j=1}^m$ is *inscribed* in the covering $\{U_i\}_{i=1}^n$ if every set U_i is a union of certain sets A_j. Among all partitions inscribed in a given covering, there exists a partition with the least possible number of elements. Each element of this minimal partition is the intersection of several sets of the form U_i and $\mathcal{C} \setminus U_i$.

If $\mathsf{A}_1 = \{A_{1,i}\}_{i=1}^{n_1}$, $\mathsf{A}_2 = \{A_{2,i}\}_{i=1}^{n_2}, \dots, \mathsf{A}_k = \{A_{k,i}\}_{i=1}^{n_k}$ is a finite set of clopen partitions, then $\mathsf{A}_1 \wedge \mathsf{A}_2 \wedge \dots \wedge \mathsf{A}_k$ is a clopen partition whose elements are nonempty sets of the form $A_{1,i_1} \cap A_{2,i_2} \cap \dots \cap A_{k,i_k}$. The partition $\mathsf{A}_1 \wedge \mathsf{A}_2 \wedge \dots \wedge \mathsf{A}_k$ is the partition with the minimum number of elements among the partitions that refine the partitions $\mathsf{A}_1, \mathsf{A}_2, \dots, \mathsf{A}_k$.

The sequence of clopen partitions

$$\{\mathcal{C}\},\ \{A_{1,i}\}_{i=1}^{n_1},\ \{A_{2,i}\}_{i=1}^{n_2}, \dots$$

is called *separating* if each partition is a refinement of the preceding partition and the diameter of partitions tends to zero.

A family of all sets $A_{k,i}$ of the separating sequence of partitions is the base of the topology of the space $\mathcal{C}$ and is naturally identified with the set of vertices of a rooted tree in which the root is identified with the set $\mathcal{C}$, while the atoms $A_{k,i}$ of the partition $\{A_{k,i}\}_{i=1}^{n_k}$ are identified with the vertices of the kth level. In this case, two vertices of adjacent levels are connected by an edge if one of the corresponding sets lies in the other. The tree constructed is called a *tree associated with the sequence of partitions.*

Proposition 6.3. *For an arbitrary totally disconnected compact metric space, there exists a separating sequence of clopen partitions such that the boundary of the tree associated with this sequence is homeomorphic to the space $\mathcal{C}$.*

Proof. Let us construct a separating sequence

$$\{\mathcal{C}\},\ \{A_{1,i}\}_{i=1}^{n_1},\ \{A_{2,i}\}_{i=1}^{n_2}, \dots$$

inductively. Suppose that the partition $\{A_{k-1,i}\}_{i=1}^{n_{k-1}}$ has already been constructed (for $k = 1$, this partition is $\{\mathcal{C}\}$).

Since $\mathcal{C}$ is a compact totally disconnected space, there exists its finite covering by clopen sets of diameter $<1/k$. Let $\{B_i\}_{i=1}^m$ be such a covering.

The next partition $\{A_{k,i}\}_{i=1}^{n_k}$ is an arbitrary clopen partition inscribed in the covering $\{B_i\}_{i=1}^m \cup \{A_{k-1,i}\}_{i=1}^{n_{k-1}}$. Each $A_{k,i}$ completely belongs to a certain B_i and to a certain $A_{k-1,j}$; hence, its diameter is less than $1/k$, and the partition $\{A_{k,i}\}_{i=1}^{n_k}$ refines the partition $\{A_{k-1,i}\}_{i=1}^{n_{k-1}}$. Thus, the sequence of partitions constructed is a separating sequence.

Each end of the tree T associated with the separating sequence of partitions corresponds to the sequence of embedded clopen sets

$$A_{1,i_1} \supseteq A_{2,i_2} \supseteq \dots .$$

The intersection $\bigcap_{k=1}^{\infty} A_{k,i_k}$ consists of a single point since the diameters of the sets A_{k,i_k} tend to zero. Conversely, for an arbitrary point $x \in \mathcal{C}$, the family of partition elements containing the point x is a sequence of embedded sets, i.e., the end of the tree T. Thus, the elements of the sets ∂T and $\mathcal{C}$ are in a natural bijective correspondence.

The fact that this correspondence is a homeomorphism easily follows from the fact that the family of all partition elements in the separating sequence is the base of the topology of the space $\mathcal{C}$. $\square$

6.3. Dynamical systems associated with the actions on trees. Let T be a spherically homogeneous rooted tree of index $\overline{m} = \{m_n\}_{n=0}^{\infty}$ and $\mathbf{X} = \{X_n\}_{n=1}^{\infty}$ be a sequence of finite sets such that T is isomorphic to the tree $T(\mathbf{X})$.

The ultrametric $d_{\overline{\lambda}}$ defines on ∂T the topology that coincides with the Tikhonov topology of the product $\mathbf{X}^\omega = \prod_{n=1}^{\infty} X_n$ of discrete spaces. Thus, the boundary of a spherically homogeneous rooted tree is homeomorphic to a Cantor set.

The group G acting by automorphisms on the rooted tree T automatically acts by homeomorphisms on its boundary. Thus, we obtain a dynamical system of the form $(G, \partial T)$ with the phase space homeomorphic to the Cantor perfect set.

Note that dynamical systems on Cantor sets play an important role in many problems of mathematics (see, for instance, [22]) and possess certain universal properties.

The dynamical systems of the form $(G, \partial T)$ have the following description.

Proposition 6.4. *Suppose that a group G acts by homeomorphisms on a totally disconnected compact metric space $\mathcal{C}$. The following conditions are equivalent.*

(i) *There exist a rooted tree T and the action of the group G on this tree by automorphisms such that the dynamical system $(G, \partial T)$ is topologically conjugate to the system $(G, \mathcal{C})$.*

(ii) *The G-orbit of each clopen set $A \subseteq \mathcal{C}$ is finite.*

Proof. The orbit of any open ball of the boundary ∂T under the action of the automorphism group of the tree T is finite since, on the set of balls, the automorphisms act as on the set of vertices of the tree. However, any clopen subset of the boundary of the tree is a union of a finite number of balls; therefore, the orbit of an arbitrary clopen set is also finite.

Conversely, assume that the orbit of every clopen set $A \subseteq \mathcal{C}$ is finite. Let $\mathsf{K} = \{K_i\}_{i=1}^{n}$ be an arbitrary clopen partition of the space $\mathcal{C}$. The orbit of each K_i is finite; therefore, the orbit of the partition K is also finite. Let $\mathsf{L} = \wedge_{g\in G}\mathsf{K}^g$. The right-hand side of this equality contains a finite number of different sets; therefore, the partition L is finite and clopen; moreover, the partition L is invariant with respect to the action of the group G.

Thus, we have proved that, for an arbitrary clopen partition of the space $\mathcal{C}$, there exists a G-invariant refinement of this partition.

Let $\mathsf{K}_0, \mathsf{K}_1, \mathsf{K}_2, \dots$ be a separating sequence of clopen partitions of the space $\mathcal{C}$ (see Proposition 6.3).

Let us construct a sequence of finite G-invariant clopen partitions $\mathsf{L}_0 = \{\mathcal{C}\}, \mathsf{L}_1, \mathsf{L}_2, \dots$ of the space $\mathcal{C}$ by choosing each L_k so that it is a refinement of the partition $\mathsf{L}_{k-1} \wedge \mathsf{K}_{k-1}$. Then, the sequence of partitions constructed also is a separating sequence. Let T be a tree associated with this sequence.

Each element $g \in G$ acts on each partition by permutations; these permutations in aggregate define the automorphism of the tree T. Accordingly, the group G acts by homeomorphisms on the boundary ∂T.

By Proposition 6.3, ∂T is homeomorphic to the space $\mathcal{C}$; this homeomorphism assigns each point $x \in \mathcal{C}$ the end consisting of the elements of partitions L_k that contain x. Therefore, the constructed action of G on ∂T and the action of G on $\mathcal{C}$ are conjugate by this homeomorphism. $\square$

In addition to the structure of a topological space, the boundary of a spherically homogeneous tree has a natural structure of a measure space.

Let m be the measure on $\mathbf{X}^\omega$ that is a product of the uniform probability measures $\{\frac{1}{m_n}, \dots, \frac{1}{m_n}\}$ on the sets X_n. Then, the measure of the boundary of the subtree T_v, where v is the vertex of the nth level, is equal to $(m_1 m_2 \dots m_n)^{-1}$. Note that the sets ∂T_v generate a σ-algebra of Borel subsets of the space $\mathbf{X}^\omega$.

The space $(\partial T, m)$ is isomorphic to the space $([0,1], l)$, where l is the Lebesgue measure. Let us identify each set X_n with the set $\{0, 1, 2, \dots, |X_n| - 1\}$; then, the isomorphism of the spaces $(\partial T(\mathbf{X}), m)$ and $([0,1], l)$ is defined by the mapping $\phi\colon T(\mathbf{X}) \to [0,1]$:

$$\phi(x_1 x_2 \dots) = \sum_{n=1}^{\infty} \frac{x_n}{m_1 m_2 \dots m_n}.$$

This mapping is surjective and measure-preserving, and all points of the interval $[0,1]$, except for a countable number, have a unique preimage. The images of the balls of the boundary ∂T_v under the mapping ϕ are intervals.

By virtue of the above identification, the dynamical systems $(G, \partial T, m)$ can also be considered as dynamical systems on the interval $[0,1]$.

The group $\operatorname{Aut} T$ acts on $(\partial T, m)$ by measure-preserving mappings. Thus, in addition to the topological dynamical systems $(G, \partial T)$, the groups acting on rooted trees also define the metric dynamical systems $(G, \partial T, m)$.

Recall that the topological dynamical system $(G, \mathcal{X})$ on the topological space $\mathcal{X}$ is called *topologically transitive* if, for any two open sets U and V of the space, there exists an element $g \in G$ such that $g(U) \cap V \neq \varnothing$. One can readily prove that, when the space $\mathcal{X}$ is metric, the topological transitivity is equivalent to the existence of a dense orbit.

The system $(G, \mathcal{X})$ is called *minimal* if each G-orbit is everywhere dense.

The dynamical system $(G, \mathcal{X}, m)$ on the space $\mathcal{X}$ with the invariant probability measure m is called *ergodic* if, for any measurable G-invariant set $A \subseteq \mathcal{X}$, either $m(A) = 0$ or $m(A) = 1$.

Proposition 6.5. *Let G be a group acting by automorphisms on a spherically homogeneous tree T. The following conditions are equivalent.*

1. *The group G is spherically transitive.*
2. *The dynamical system $(G, \partial T)$ is minimal.*
3. *The dynamical system $(G, \partial T)$ is topologically transitive.*
4. *The dynamical system $(G, \partial T, m)$ is ergodic.*
5. *The measure m is a unique σ-additive probabilistic G-invariant measure on ∂T.*

Proof. The equivalence of the first three conditions directly follows from the definitions.

Suppose that G is spherically transitive and $A \subseteq \partial T$ is a G-invariant measurable set. By the Lebesgue density theorem, for an arbitrary point $x \in \partial T$, the limit

$$\lim_{\delta \to 0} \frac{m(A \cap B(x,\delta))}{m(B(x,\delta))},$$

where $B(x,\delta) \subset \partial T$ is a ball of the ultrametric space $(\partial T, d)$ of radius δ with the center at the point x, is equal either to zero or to unity.

In the first case, for any $\epsilon > 0$, there exists $\delta > 0$ such that $m(A \cap B(x,\delta)) < \epsilon m(B(x,\delta))$. Since the group G is transitive on the levels, the balls $B(x^g, \delta)$ cover the whole boundary ∂T. Note that two balls of radius δ either coincide or do not intersect. Hence, $m(A) < \epsilon$ for any $\epsilon > 0$, i.e., $m(A) = 0$. The fact that $m(A) = 1$ when the limit is equal to unity is proved similarly.

Suppose that the group G is not spherically transitive. In this case, the set of vertices of a certain level can be partitioned into two nonempty G-invariant sets A and B. Then, the set

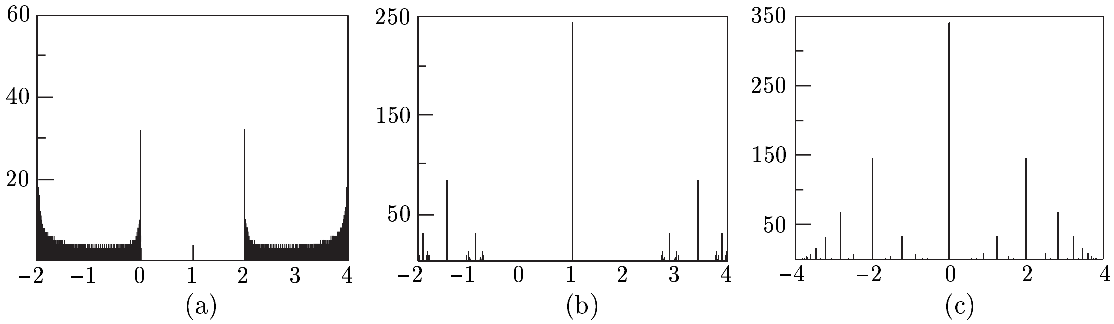


Fig. 27. The spectra of the dynamical systems defined by finite automata

of ends passing through the vertices of the set A is G-invariant, and $0 < m(A) < 1$; hence, G is nonergodic.

It remains to prove that the spherical transitivity is equivalent to the uniqueness of the G-invariant measure on ∂T.

It is clear that the σ-algebra of Borel sets on ∂T is generated by the family of the sets $\{\partial T_v : v \in V(T)\}$. However, if the group G is spherically transitive, then, for any G-invariant probability measure l on ∂T, we have $l(\partial T_v) = m(\partial T_v)$; therefore, $l = m$.

Conversely, if the group G is not spherically transitive, then it is nonergodic and, for arbitrary G-invariant sets $A, B \subset \partial T$ such that $A \cap B = \varnothing$ and $0 < m(A) < 1$, the conditional measures $m(\,\cdot\,|A)$ and $m(\,\cdot\,|B)$ are invariant and different from m. □

Let $S = S^{-1}$ be a finite set of invertible transformations of the space $(\mathcal{X}, m)$ that preserve the class of measure m. Let G be the group of transformations of the space $(\mathcal{X}, m)$ generated by the set S. We obtain a natural unitary representation π of the group G in the Hilbert space $L^2(\mathcal{X}, m)$ defined by the equality

$$(\pi(g)f)(x) = \sqrt{p_g(x)}\, f(g^{-1}x),$$

where $g \in G$ and $p_g(x) = dgm(x)/dm(x)$ is the Radon–Nikodým derivative.

The *spectrum of the dynamical system* $(S, \mathcal{X}, m)$ is defined as the spectrum of the operator

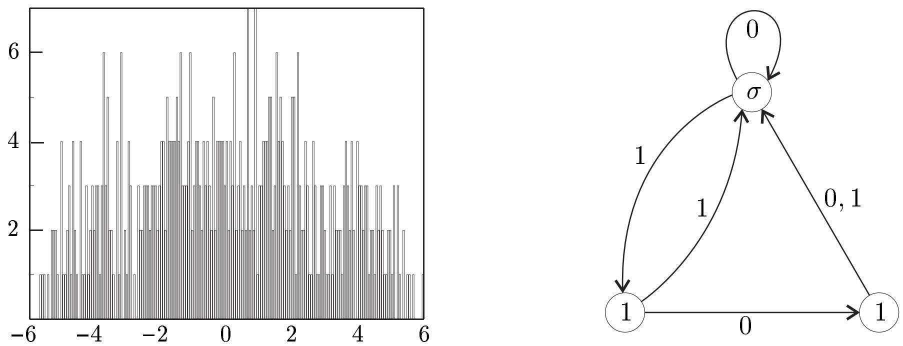
$$H_\pi = \frac{1}{|S|} \sum_{s \in S} \pi(s)$$

on the Hilbert space $L^2(\mathcal{X}, m)$.

Definition 6.3. Let A be a noninitial invertible synchronous automaton. For any internal state q of this automaton, the initial automaton A_q defines a measure-preserving transformation of the space X^ω. Let S be the set of all transformations defined by the automata A_q together with their inverses. The set S generates a group $G(A)$ *defined by the automaton* A.

The spectrum of the dynamical system (S, X^ω, m) is called the *spectrum of the dynamical system defined by the automaton* A.

The spectra of the dynamical systems defined by finite synchronous automata are very diverse even for the automata with a small number of states. Examples are given in Fig. 27 by the histograms of the spectral measures of the dynamical systems defined by some of these automata. (Note that this figure demonstrates the histograms of the spectra that are not normalized by the number of Hecke-type generating operators.)



(a) (b)

Fig. 28

Figure 27a represents the histogram of the spectrum of the dynamical system defined by the Grigorchuk group [71]; Figure 27b, the histogram corresponding to the Gupta–Sidki 3-group [27]; and Figure 27c corresponds to the lamplighter group (see [7, 30] for more detail). Note that, in the first case, the histograms converge to a smooth curve, in the second, they give a Cantor set in the limit, and in the third case, the limit spectral measure proves to be discrete and concentrated on a countable everywhere dense set.

The fractal character of the spectra associated with automata is displayed by the histogram depicted on Fig. 28a, which corresponds to the three-state automaton depicted on Fig. 28b.

6.4. Action of cyclic groups. Consider the dynamical systems on the boundary of a rooted tree that are induced by the action of one automorphism.

Let $f \in \operatorname{Aut} T$ be an automorphism of a rooted tree T. Consider a graph $T/\langle f\rangle$ whose vertices and edges are, respectively, the orbits of the action of the cyclic group $\langle f\rangle$ on the set of vertices and on the set of edges of the tree T. If $\overline{v}_1$ and $\overline{v}_2$ are two vertices of the graph $T/\langle f\rangle$, then they are connected by an edge $\overline{e}$ if and only if there exists an edge e in the orbit $\overline{e}$ that connects certain two vertices v_1 and v_2 such that $v_1 \in \overline{v}_1$ and $v_2 \in \overline{v}_2$.

It can be readily proved that the *orbit graph* $T/\langle f\rangle$ is a tree. Let us transform this tree to a rooted tree by choosing a one-point orbit of the root of the tree T as the root. Let $\pi\colon T \to T/\langle f\rangle$ be the canonical projection.

The projection π translates the levels of the tree T to the corresponding levels of the tree $T/\langle f\rangle$ and is a morphism of trees.

The *orbital type* T_f of the automorphism f is the tree $T/\langle f\rangle$ in which each vertex is labeled by a natural number equal to the cardinality of the corresponding orbit. Two labeled rooted trees are called isomorphic if there exists a label-preserving isomorphism between these trees.

The following criterion is valid (see [67, 66]).

Theorem 6.6. *Two automorphisms of the rooted tree T are conjugate in the group $\operatorname{Aut} T$ if and only if their orbital types are isomorphic.*

The orbital types (i.e., the graphs of the orbits) carry information also about the action of the automorphism f on the boundary of the tree.

Proposition 6.7. *Let γ_1 and γ_2 be two ends of the tree T and $\pi\colon T \to T/\langle f\rangle$ be the canonical projection. If $\pi(\gamma_1) = \pi(\gamma_2)$, then the closures of the f-orbits of the ends γ_1 and γ_2 coincide; otherwise, they do not intersect.*

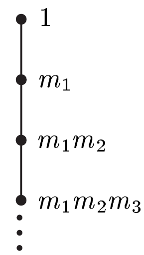


Fig. 29

From the viewpoint of the metric dynamics, the boundary of the tree of orbits has the following interpretation that is implied by Propositions 6.7 and 6.5.

Proposition 6.8. *The ergodic invariant probability measures of a dynamical system* $(\partial T, f)$ *are in one-to-one correspondence with the points of the boundary of the orbit tree* $T/\langle f\rangle$.

Thus, the dynamical system $(\partial T, f)$ is transitive (minimal) if and only if the boundary ∂T_f is one-point. In this case, the orbital type of the automorphism f acting on the tree with the spherical index $\{m_n\}_{n=1}^{\infty}$ has the form shown in Fig. 29 since the transitivity of the system is equivalent to the spherical transitivity.

An automorphism $f \in \operatorname{Aut} T$ is called *spherically transitive* if the cyclic group $\langle f\rangle$ is spherically transitive.

By Proposition 6.7 (see also [9]), the following assertion is valid.

Proposition 6.9. *Two arbitrary spherically transitive automorphisms of a rooted tree are conjugate.*

An example of spherically transitive automorphisms of a rooted tree is given by the adding machine, i.e., by the automorphism defined by the automaton depicted in Fig. 7.

For irregular spherically homogeneous trees, the adding machines are defined as follows.

Let T be a spherically homogeneous tree of spherical index $\overline{m} = \{m_1, m_2, \dots\}$. Consider the inverse spectrum of finite cyclic groups

$$\mathbb{Z}/1\mathbb{Z} \leftarrow \mathbb{Z}/m_1\mathbb{Z} \leftarrow \mathbb{Z}/m_1m_2\mathbb{Z} \leftarrow \mathbb{Z}/m_1m_2m_3\mathbb{Z} \leftarrow \dots, \tag{10}$$

where $\mathbb{Z}/m_1m_2\dots m_k\mathbb{Z}$ are the groups of residues modulo $m_1m_2\dots m_k$, while the morphisms are canonical (i.e., such that the image of the residue $\overline{1}$ of one group is the residue $\overline{1}$ of the other). Consider the inverse limit

$$\mathbb{Z}_{\overline{m}} = \lim_{\leftarrow} \mathbb{Z}/m_1m_2\dots m_k\mathbb{Z}.$$

In the case of a regular p-tree (a tree with the spherical index $(p, p, \dots)$, where p is a prime number), this limit is isomorphic to the group of p-adic integers.

Consider a family of all elements of the groups of the inverse spectrum (10). Let us connect two elements $a \in \mathbb{Z}/m_1m_2\dots m_k\mathbb{Z}$ and $b \in \mathbb{Z}/m_1m_2\dots m_{k+1}\mathbb{Z}$ by an edge if the first element is the image of the second under the action of a homomorphism from the chain (10). The obtained graph with the root $0 \in \mathbb{Z}_1$ is a spherically homogeneous rooted tree T' with the spherical index $\overline{m}$ and thereby is isomorphic to the tree T. In this case, the boundary of the obtained tree T' is identified with the group $\mathbb{Z}_{\overline{m}}$.

The mapping $f\colon a \mapsto a + \overline{1}$ is an automorphism of the rooted tree T'. It is clear that the automorphism f is spherically transitive. It is this automorphism that is called the adding machine. The adding machines are important examples of dynamical systems on the Cantor set that arise in one-dimensional dynamics, in the theory of Lyapunov stable sets, and in other fields (see [43, 8, 12, 9]).

Let $\mathcal{X}$ be a topological space and $f\colon \mathcal{X} \to \mathcal{X}$ be a continuous function. A number $\lambda \in \mathbb{C}$ is called the *eigenvalue for the eigenfunction* $\phi \in C(\mathcal{X}, \mathbb{C})$, $\phi \neq 0$, of the dynamical system $(\mathcal{X}, f)$ if $\phi(f(x)) = \lambda\phi(x)$ for all $x \in \mathcal{X}$.

The dynamical system $(\mathcal{X}, f)$ is called a system *with topologically discrete spectrum* if a closed linear span, in $C(\mathcal{X}, \mathbb{C})$, of the set of eigenfunctions of the system $(\mathcal{X}, f)$ coincides with $C(\mathcal{X}, \mathbb{C})$.

The set of eigenvalues of the adding machine is described as follows.

Proposition 6.10 [12]**.** *Let T be a spherically homogeneous tree, $\overline{m} = \{m_i\}_{i=1}^{\infty}$ be its spherical index, and $f \in \operatorname{Aut} T$ be a spherically transitive automorphism of the tree T.*

Then, the dynamical system $(\partial T, f)$ has a topologically discrete spectrum, and the set of its eigenvalues is equal to

$$\left\{e^{2\pi i k/m_1 m_2 \ldots m_n}\colon\ 0 \le k < m_1 m_2 \ldots m_n,\ n > 0\right\}.$$

By using the criterion of conjugacy of minimal dynamical systems with topologically discrete spectra, the following criterion of topological conjugacy of two adding machines was proved in [12].

Proposition 6.11. *Let T_1 and T_2 be spherically homogeneous rooted trees with spherical indices $(m_1, m_2, \ldots)$ and $(k_1, k_2, \ldots)$, respectively. If the automorphisms $f_1 \in \operatorname{Aut} T_1$ and $f_2 \in \operatorname{Aut} T_2$ are spherically transitive, then the dynamical systems $(\partial T_1, f_1)$ and $(\partial T_2, f_2)$ are topologically conjugate if and only if, for any prime number p, the sum of maximal exponents in which p divides the numbers m_i (this number may be equal to infinity) is equal to a similar sum for the numbers k_i.*

6.5. Cycles of synchronous automatic transformations. Recall that a finite subset $\{x_1, x_2, \ldots, x_n\}$ of the set A is called the *cycle* of transformation $f\colon A \to A$ if

$$f(x_i) = x_{i+1}, \quad 1 \le i \le n-1, \qquad f(x_n) = x_1;$$

here, the number n is called the *cycle length*, and the elements x_i are called *cyclic elements of order* n for the transformation f. The symbol $\mathrm{Cycl}(f)$ denotes the set of orders of all cyclic elements of f.

Let $\mathfrak{F}$ be a certain class of self-mappings of A. The main problems concerning the cycle structure of the mappings from $\mathfrak{F}$ can be formulated as follows.

1. What natural numbers may serve as the cycle lengths for the transformations from $\mathfrak{F}$?
2. What sets of natural numbers may serve as the sets of all cycle lengths for the transformations from $\mathfrak{F}$?

In certain cases, the answers to these questions are known. For example, according to the Sharkovskii theorem [91] (see also [3, Theorem 2.1.1]), if $\mathfrak{F}$ is a set of all continuous transformations of the real interval $[0, 1]$, then

(1) any number may serve as the cycle length for a transformation from $\mathfrak{F}$;

(2) the subset $M \subseteq \mathbb{N}$ has the form $\mathrm{Cycl}(f)$, $f \in \mathfrak{F}$, only if it consists of all numbers less than a certain prescribed number when $\mathbb{N}$ is arranged in the Sharkovskii order,

$$3 > 5 > 7 > \ldots > 2\cdot 3 > 2\cdot 5 > \ldots > 2^2\cdot 3 > 2^2\cdot 5 > \ldots > 2^3 > 2^2 > 2 > 1.$$

The case of polynomial mappings of various fields and rings was considered in Narkiewicz's book [44]. Here, we propose a solution to this problem for synchronous automatic transformations over a finite alphabet X that act on the set X^{ω}.

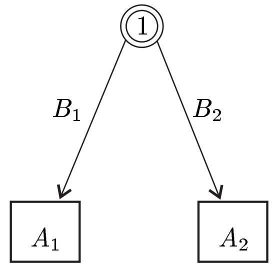


Fig. 30

Denote by the symbol E_X the set of all natural numbers whose decomposition involves only prime factors that are not greater than $|X|$. If $|X| = 2$, then the set E_X consists only of powers of two, whereas, if $|X| = 3$ or $|X| = 4$, then $E_X = \{2^\alpha \cdot 3^\beta : \alpha, \beta = 0, 1, 2 \ldots\}$.

Theorem 6.12. *A number n is the cycle length of a certain synchronous automatic transformation $f: X^\omega \to X^\omega$ if and only if it is contained in E_X.*

The answer is the same for the case of invertible synchronous automatic transformations. The answer to the question about the existence of cycle lengths is given by the following theorem.

Theorem 6.13. *For any subset $M \subseteq E_X$, there exists a synchronous automatic bijective transformation $f: X^\omega \to X^\omega$ such that* $\operatorname{Cycl} f = M$.

Here, we present the sketch of the proof (the full proof is given in [60]).

If M_1 and M_2 are the sets of cycle lengths on X^ω of certain synchronous automatic transformations, then the set $M_1 \cup M_2$ can also be represented as $\operatorname{Cycl} f$, where f is a synchronous automatic transformation. Indeed, if A_1 and A_2 are the automata defining the corresponding transformations for the sets M_1 and M_2, then the set of cycle lengths of the action of the automaton depicted in Fig. 30 on the space X^ω will be equal to $M_1 \cup M_2$. In this figure, the arrows starting from the initial state end at the initial states of the automata A_1 and A_2, and B_1, B_2 is the partition of the alphabet into nonempty sets.

A subset $M \subseteq E_X$ is called a D-subset if there exists a number $c \in M$ such that all elements of M are divisible by c.

Lemma 6.14. *For an arbitrary D-subset $M \subseteq E_X$, there exists a synchronous automatic bijective transformation $f: X^\omega \to X^\omega$ such that* $\operatorname{Cycl} f = M$.

Lemma 6.14 implies that the union of a finite number of D-sets can also be represented as $\operatorname{Cycl} f$ for a certain synchronous automatic transformation $f: X^\omega \to X^\omega$.

Now, it remains to prove that an arbitrary set $M \subseteq E_X$ can be represented as a union of a finite number of D-sets. This is equivalent to the fact that, in the set E_X that is partially ordered by the divisibility relation, each antichain (i.e., the set of pairwise noncomparable elements) is finite. However, the set E_X ordered by the divisibility relation is order isomorphic to the Cartesian product $\mathbb{N}^k$ with the order

$$(n_1, n_2, \ldots, n_k) \le (m_1, m_2, \ldots, m_k) \quad \Leftrightarrow \quad n_i \le m_i, \quad 1 \le i \le k,$$

where k is the number of prime numbers no greater than $|X|$. The finiteness of the antichains in this partially ordered set is readily proved by induction on k.

The results obtained can be interpreted in terms of transformations of a ring of p-adic integer numbers as follows.

Let $\mathbb{Z}_p$ be a ring of p-adic integers with the natural metric

$$\rho(u,v) = v_p(u-v),$$

where v_p is the p-adic norm. In other words, if $u = u_0u_1\dots$ and $v = v_0v_1\dots$ are the canonical representations of the numbers u and v ($0 \le v_i \le p-1$), then $\rho(u,v) = (\frac{1}{p})^k$, where k is the length of the common beginning of the sequences $u_0u_1\dots$, $v_0v_1\dots$. The mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is called *nonexpansive* if $\rho(f(u),f(v)) \le \rho(u,v)$ for any $u,v \in \mathbb{Z}_p$. A mapping is nonexpansive if and only if it defines a synchronous automatic transformation of infinite sequences over the alphabet $\{0,1,\dots,p-1\}$. Denote by E_p the set of natural numbers whose canonical decompositions contain only prime factors no greater than p. Theorems 6.12 and 6.13 imply the following corollary.

Corollary 6.15. *A number n is the cycle length of a certain nonexpansive mapping of the metric space $(\mathbb{Z}_p,\rho)$ if and only if $n \in E_p$.*

For any subset $M \subseteq E_p$, there exists a nonexpansive mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ such that $\operatorname{Cycl} f = M$.

Thus, there does not exist any order similar to the Sharkovskii order even in the case of nonexpansive mappings of a ring of p-adic integers.

6.6. The Lyapunov stability. Let $\mathcal{X}$ be a locally compact and locally connected metric space and $f\colon \mathcal{X} \to \mathcal{X}$ be a continuous mapping. A set $A \subseteq \mathcal{X}$ is *invariant* with respect to f if $f(A) \subseteq A$. A compact invariant subset $A \subseteq \mathcal{X}$ is called *Lyapunov stable* with respect to f if, for any neighborhood U of the set A, there exists a neighborhood V of the set A such that $V \subseteq U$ and $f^n(V) \subseteq U$ for any $n \ge 0$.

The following definition naturally extends the concept of Lyapunov stability to the group actions.

Definition 6.4. Let G be a group acting on $\mathcal{X}$ by homeomorphisms and $H \le G$ be a subsemigroup generating G as a group. A compact G-invariant set $A \subseteq \mathcal{X}$ is called *Lyapunov H-stable* if, for any open neighborhood $U \supset A$, there exists an open neighborhood $V \supset A$ such that $V \subseteq U$ and $h(V) \subseteq U$ for any $h \in H$.

It was proved in [12, 11] that the action of a dynamical system on the connected components of a Lyapunov stable transitive set is topologically conjugate to the action of the adding machine on the boundary of a tree. Below, we extend this result to the case of a group action.

Let $A \subseteq \mathcal{X}$ be a compact G-invariant subset and $\mathcal{C}$ be the set of its connected components. We equip the set $\mathcal{C}$ with the topology of a quotient space. This topology coincides with the topology defined by the Hausdorff metric on the subsets of the space $\mathcal{X}$. Then, $\mathcal{C}$ is a totally disconnected compact metric space. Each homeomorphism f of the space $\mathcal{X}$ such that A is invariant induces on $\mathcal{C}$ a homeomorphism $\tilde{f}\colon \mathcal{C} \to \mathcal{C}$ that is consistent with the projection $\pi\colon A \to \mathcal{C}$. If the action of the group G on A is topologically transitive or minimal, then, respectively, the induced action of G on $\mathcal{C}$ is also topologically transitive or minimal; in these cases, the set $\mathcal{C}$ either is finite or does not have isolated points and thereby is homeomorphic to a Cantor set.

Theorem 6.16. *Let $\mathcal{X}$ be a locally connected and locally compact metric space, G be a group acting on $\mathcal{X}$ by homeomorphisms, and $A \subseteq \mathcal{X}$ be a compact G-invariant set with an infinite number of connected components.*

If $H \le G$ is a subsemigroup of the group G that generates G as a group and the set A is Lyapunov H-stable, then there exists an action of the group G by automorphisms on a certain infinite rooted tree T such that the induced action of G on the space $\mathcal{C}$ of connected components of the set A is topologically conjugate to the action of G on ∂T.

We will use without proof the following auxiliary statement from [12, Proposition 5.3].

Lemma 6.17. *Let $\mathcal{X}$ be a locally connected metric space and $A \subset \mathcal{X}$ be a compact subset. Then, an arbitrary open neighborhood $U \supset A$ contains an open neighborhood $V \supset A$ that is a disjoint union of a finite number of nonempty connected open sets.*

Proof of Theorem 6.16. Without loss of generality, we can assume that H contains the identity of the group G. Let $\pi\colon A \to \mathcal{C}$ be the natural projection.

For an arbitrary subset $B \subseteq \mathcal{X}$, we denote $\widetilde{B} = \pi(A \cap B)$ and, for every homeomorphism $h \in G$, denote by $\tilde{h}$ its induced action on $\mathcal{C}$.

By Proposition 6.4, to prove the conjugacy of the dynamical system $(G, \mathcal{C})$ to the dynamical system $(G, \partial T)$ for a certain action of the group G on a rooted tree, it suffices to prove that the orbit of an arbitrary clopen subset K of the space $\mathcal{C}$ under the action of G is finite.

The sets $\pi^{-1}(K)$ and $\pi^{-1}(\mathcal{C} \setminus K)$ are closed in the relative topology of the set $A \subset \mathcal{X}$ and, since A is closed, they are also closed in the topology of the space $\mathcal{X}$. By the normality of the metric spaces, there exist nonintersecting open neighborhoods $U' \supset \pi^{-1}(K)$ and $U'' \supset \pi^{-1}(\mathcal{C} \setminus K)$ in $\mathcal{X}$. Let $U_0 = U' \cup U''$ be a neighborhood of the set A. By Lemma 6.17, there exists a neighborhood $U \subseteq U_0$ of the set A that can be represented as a disjoint union $U = \bigcup_{i=1}^{n} U_i$ of open connected sets. Then, possibly after renumbering, $U' \supseteq U_1 \cup U_2 \cup \ldots \cup U_r$ and $U'' \supseteq U_{r+1} \cup U_{r+2} \cup \ldots \cup U_n$ for a certain r.

By the definition of stability, there exists a neighborhood $V \supset A$ such that $h(V) \subseteq U$ for all $h \in H$. Here, by Lemma 6.17, we can assume that there exists a representation of V as a disjoint union of open connected sets, $V = \bigcup_{j=1}^{m} V_j$. Then, each set $\widetilde{U}_i$ is a union of certain sets $\widetilde{V}_j$.

Note that $\{\widetilde{U}_i\}_{i=1}^{n}$ and $\{\widetilde{V}_j\}_{j=1}^{m}$ are clopen partitions of the space $\mathcal{C}$ and $K = \bigcup_{i=1}^{r} \widetilde{U}_i$.

For every $h \in H$, the set $h(V_j)$ is connected and therefore belongs only to one of the subsets U_i.

Let, for certain V_j and $h \in H$, the set $h(V_j)$ have a nonempty intersection with two sets V_{j_1} and V_{j_2}. Then, for any $f \in H$, the set $f(h(V_j) \cup V_{j_1} \cup V_{j_2})$ completely belongs to the set U_i that contains $f(h(V_j))$ since all U_i are disjoint. Let us replace the pair V_{j_1}, V_{j_2} in the union $V = \bigcup_{j=1}^{m} V_j$ by the open set $W_{j_1,j_2} = V_{j_1} \cup V_{j_2}$. In the new family of open sets, just as in the original one, the image of every element under the action of an arbitrary homeomorphism $h \in H$ completely belongs to one of the sets U_i.

Proceeding in this way, we reduce the number of sets of the original family $\{V_i\}$ and finally arrive at a family of disjoint open sets $\{W_i\}_{i=1}^{k}$ such that $V = \bigcup_{i=1}^{k} W_i$ and, for arbitrary $h \in H$ and W_i, the set $h(W_i)$ may intersect at most one set W_j. Therefore, for any $h \in H$, the set $\tilde{h}(\widetilde{W}_i)$ is contained exactly in one set $\widetilde{W}_j$. However, since there is a finite number of sets $\widetilde{W}_i$ and $\tilde{h}$ is a homeomorphism, this implies that $\tilde{h}$ permutes the sets $\widetilde{W}_i$ and, since H generates the group G, each homeomorphism $g \in G$ also permutes the sets $\widetilde{W}_i$; i.e., the clopen partition $\{\widetilde{W}_i\}_{i=1}^{k}$ is invariant under the action of the group G.

Since $K = \bigcup_{i=1}^{r} \widetilde{U}_i$, K is a union of certain sets $\widetilde{W}_j$; therefore, the orbit of K under the action of G is finite. $\square$

By Proposition 6.5, we obtain the following assertion.

Corollary 6.18. *Under the hypotheses of Theorem* 6.16, *the property of spherical transitivity of the action of the group G on the tree T and the properties of minimality and transitivity of the dynamical system $(G, \mathcal{C})$ are equivalent.*

6.7. The Schreier graphs. Recall that a *graph* Γ is defined by the *vertex set* $V(\Gamma)$, the *edge set* $E(\Gamma)$, the functions $\alpha, \omega\colon E(\Gamma) \to V(\Gamma)$ (the vertices $\alpha(e)$ and $\omega(e)$ are called the *beginning and end* of the edge e), and the involution $e \mapsto \overline{e}$ such that $\alpha(\overline{e}) = \omega(e)$. The (*edge-*)*labeled graph* is a graph in which each edge is assigned a *label*—an element of a certain *label set* S.

The *morphism of graphs* $f\colon \Gamma_1 \to \Gamma_2$ is a pair of mappings $f_{\mathrm{v}}\colon V(\Gamma_1) \to V(\Gamma_2)$, $f_{\mathrm{e}}\colon E(\Gamma_1) \to E(\Gamma_2)$ that satisfy the relations

$$\alpha(f_{\mathrm{e}}(y)) = f_{\mathrm{v}}(\alpha(y)), \qquad \overline{f_{\mathrm{e}}(y)} = f_{\mathrm{e}}(\overline{y})$$

for all $y \in E(\Gamma_1)$. The *morphism of labeled graphs* is the morphism of graphs that preserves the labels of edges.

Let G be a finitely generated group with the system S of generators such that $1 \notin S$ and $S = S^{-1}$ that acts faithfully on a certain set M. Then, the *action graph* $\Gamma^*(G, S, M)$ of the group G on M is a labeled graph with the set of vertices M and the set of edges $M \times S$. Each edge (x, s), $x \in M$, $s \in S$, is labeled by the generator s, and the functions α and ω and the involution are defined by the relations

$$\alpha((x,s)) = x, \qquad \omega((x,s)) = x^s, \qquad \overline{(x,s)} = (x^s, s^{-1}),$$

where $x \in M$ and $s \in S$.

It is obvious that the action graph uniquely determines the action of the generators of the group G on the set M and, hence, it defines the group G due to the faithfulness of the action.

The *Schreier graph* $\Gamma(G, S)$ of the group G on the set M is the graph obtained from the action graph $\Gamma^*(G, S)$ by erasing the labels.

Let $T(\mathbf{X})$ be a spherically homogeneous tree constructed by the sequence of finite sets $\mathbf{X} = \{X_1, X_2, \dots\}$.

Let G be a spherically transitive automorphism group of the tree $T(\mathbf{X})$ generated by a finite set S satisfying the conditions imposed above, and let $\Gamma^*(G, S)$ and $\Gamma(G, S)$ be, respectively, the action graph of this group on $\mathbf{X}^\omega$ and the Schreier graph. The graphs $\Gamma^*(G, S)$ and $\Gamma(G, S)$ consist of an infinite number of countable connected components. The vertex set of each connected component is the action orbit of G on $\mathbf{X}^\omega$, while the corresponding component is the action graph (the Schreier graph) of the group on this orbit.

Denote by $\Gamma_n^*(G, S)$ the action graph of the group G at the level $\mathbf{X}^n$ and by $\Gamma_n(G, S)$ the corresponding Schreier graph.

It directly follows from the definitions and properties of the actions of automata groups on the words that the mapping $\pi_n\colon \mathbf{X}^n \to \mathbf{X}^{n-1}$ (which eliminates the last coordinate), considered as the mapping of a vertex set, is naturally extended to the morphism of labeled graphs $\pi_n\colon \Gamma_n^*(G, S) \to \Gamma_{n-1}^*(G, S)$:

$$\pi_n(w, s) = (\pi_n(w), s).$$

Thus, we obtain the following inverse spectrum of the graphs:

$$\Gamma_0^*(G, S) \leftarrow \Gamma_1^*(G, S) \leftarrow \Gamma_2^*(G, S) \leftarrow \dots.$$

Proposition 6.19. *The graphs $\Gamma^*(G, S)$ and $\Gamma(G, S)$ of action of the group G on the boundary $\mathbf{X}^\omega$ are the inverse limits of the corresponding inverse spectra of the finite graphs $\Gamma_n^*(G, S)$ and $\Gamma_n(G, S)$.*

Hence, the action graph on the boundary is profinite (see [57] for profinite graphs).

Definition 6.5. Let G be a spherically transitive countable automorphism group of a tree $T(\mathbf{X})$. A point $w \in \mathbf{X}^\omega$ of the boundary of the tree is called a *generic point* with respect to $g \in G$ if either $w^g \neq w$ or there exists a neighborhood U of the point w that completely consists of fixed points of the automorphism g.

A point $w \in \mathbf{X}^\omega$ is a *generic point* with respect to the group G if it is generic with respect to each element of G.

Proposition 6.20. *For any countable group G of automorphisms of the tree $T(\mathbf{X})$, almost all points of the boundary $\mathbf{X}^\omega$ in the sense of the Baire category are generic points with respect to G.*

Proof. The set of nongeneric points with respect to a tree automorphism g is a nowhere dense set. Hence, the set of all nongeneric points with respect to a countable group is a union of a countable number of nowhere dense sets. □

A *finite path* in the graph Γ is a sequence of its edges $e_1, e_2, \dots, e_n$, where $\omega(e_i) = \alpha(e_{i+1})$ for every $1 \le i \le n-1$. The vertex $\alpha(e_1)$ is called the *beginning of the path*, while the vertex $\omega(e_n)$ is its *end*. The number n is called the *path length*. An infinite path is defined similarly.

For a graph Γ, vertex $v \in V(\Gamma)$, and $r \in \mathbb{N}$, a *ball $B(v,r)$ of radius r with the center at the point v* is defined as a subgraph with the set of edges that belong to all paths that start at v and have the length at most r, while the set of vertices consists of v and all vertices that are the ends of these paths.

Two graphs Γ_1 and Γ_2 are called *locally isomorphic* if, for any vertex v of one of the graphs and $r \in \mathbb{N}$, there exists a vertex u of the other graph such that the graphs $B(v,r)$ and $B(u,r)$ are isomorphic.

Generic points with respect to the action of a finitely generated automorphism group of a tree have the action graphs that possess typical properties. More precisely, the following assertion holds.

Proposition 6.21. *Let G be a finitely generated spherically transitive group of automorphisms of the tree $T(\mathbf{X})$. If $w \in \mathbf{X}^\omega$ is a generic point with respect to G and $r \in \mathbb{N}$, then, in each connected component of the graph $\Gamma^*(G,S)$ (i.e., in each orbit), there exists a vertex v such that the ball $B(w,r)$ is isomorphic to the ball $B(v,r)$. Any two action graphs on the orbits of generic points are locally isomorphic.*

For any labeled graph Γ that is locally isomorphic to the action graph of the group G on a certain orbit of a generic point, there exists an orbit of the group G with the action graph isomorphic to the graph Γ.

Thus, almost all (in the sense of the Baire category) Schreier graphs of the action orbits of the group G on the boundary are locally isomorphic.

One can easily prove that a similar assertion is also valid in the sense of measure. Namely, the following proposition holds.

Proposition 6.22. *Let G be a finitely generated spherically transitive group of automorphisms of the tree $T(\mathbf{X})$ and m be a G-invariant probability measure on the boundary. Almost all (in the sense of the measure m) points of the boundary $\mathbf{X}^\omega$ have locally isomorphic action graphs of the group G on their orbits.*

It would be interesting to construct an example of the group in which the Schreier graphs of the orbits that are typical in the sense of the Baire category (the orbits of generic points) are different from the Schreier graphs of the orbits that are typical in the sense of measure.

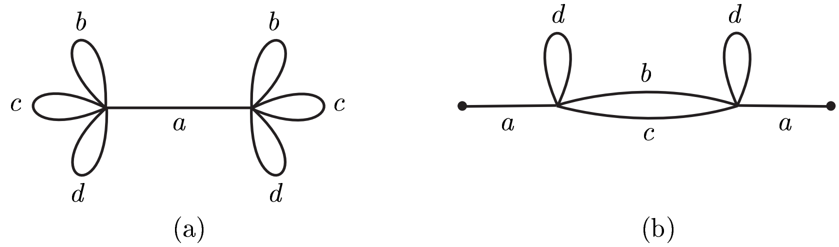


Fig. 31

When the automorphism group of a tree is defined by a synchronous finite automaton, the Schreier graphs have a specific self-similar (fractal) structure. Moreover, frequently, the Schreier graphs of orbits are the inductive limits of the Schreier graphs of the group action on the levels.

The first examples of the study of the Schreier graphs of action orbits for groups defined by finite synchronous automata are the works [7, 30], in which, in particular, the spectrum of a discrete Laplace operator was investigated on such graphs.

A *discrete Laplace operator* on the graph Γ is the operator $\Delta = 1 - M$ on $\ell^2(V(\Gamma))$, where

$$(Mf)(v) = \frac{1}{\deg v} \sum_{e \in E(\Gamma),\ \alpha(e)=v} f(\omega(e)),$$

$f \in \ell^2(V(\Gamma))$, $v \in V(\Gamma)$, and $\deg v$ is the valence of the vertex v. The operator M is called the *Markov operator*, or the *random walk operator*.

In [7], it was proved, in particular, that the spectrum of the Laplace operator for the group defined by the automaton depicted in Fig. 9 is a union of a Cantor set and a certain countable set of isolated points and can be described as the closure of the set

$$\left\{4,\ 1,\ 1 \pm \sqrt{6},\ 1 \pm \sqrt{6 \pm \sqrt{6}},\ 1 \pm \sqrt{6 \pm \sqrt{6 \pm \sqrt{6}}}, \dots\right\}.$$

In addition, in [7], the spectra of certain other graphs were found for the actions of groups defined by finite automata.

As was mentioned in the introduction, the second work in this direction is [30]. In this work, the lamplighter group $\mathbb{Z} \wr \mathbb{Z}_2$ was realized as a group defined by the automaton depicted in Fig. 6. It was proved that, in this case, the Schreier graph of almost every orbit is isomorphic to the Cayley graph of the group, the spectrum of the operator M is the interval $[-1, 1]$, and the spectral measure is a discrete measure concentrated on a countable dense subset of the interval $[0, 1]$. This is the first example of the group with such an unusual spectral property.

To calculate the spectra of the groups defined by automata, the method of approximation by finite graphs is used, where the Schreier graphs of actions on the levels of a tree serve as such finite graphs. This method was described in [7, 30, 29]. Here, the property of self-similarity of a group and its action as well as the substitutional character of the corresponding graph play an important role.

For example, the graphs of action of the Grigorchuk 2-group (the group defined by the automaton shown in Fig. 14) on the levels of a tree can be obtained as follows. Let Γ_1 be a labeled graph depicted in Fig. 31a. It is clear that this graph is the graph of action of the Grigorchuk group on the first level of the tree. The graph Γ_n is obtained from the graph Γ_{n-1} by simultaneous replacement of all labels b by the labels d, labels c by b, d by c, and all edges labeled by a by the graph depicted in Fig. 31b; here, the ends of the original edge correspond to the marked vertices of the graph. The graphs Γ_n thus obtained are the graphs of action of the Grigorchuk group on the levels of the tree.

The graphs of action on the levels of the tree for the group defined by the automaton depicted in Fig. 9 and the Gupta–Sidki group are constructed in a similar way [27].

6.8. Orbits of group actions on the boundary of a tree. Let G be a countable group acting by automorphisms on the rooted tree T. Denote by E_G the equivalence relation on the boundary ∂T such that two points are equivalent if and only if they belong to the same G-orbit. The relation E_G, considered as a subset of the set $\partial T \times \partial T$, is Borel. More precisely, it is an F_σ-set, i.e., the union of a countable number of closed sets.

Conversely, by the Feldman–Moore theorem [20], for an arbitrary countable (i.e., with countable classes) Borel relation E on a standard Borel space $\mathcal{X}$, there exists a countable group G of Borel automorphisms of the space $\mathcal{X}$ such that $E = E_G$, where E_G, as above, is the relation "to belong to the same G-orbit."

Recall certain definitions of the theory of Borel equivalence relations and point out specific features associated with the actions of groups on rooted trees.

Definition 6.6. A countable Borel equivalence relation $E \subseteq \mathcal{X} \times \mathcal{X}$ on the space with the measure $(\mathcal{X}, m)$ *preserves the measure* m if, for any measurable set A and any measurable injective mapping $f\colon A \to \mathcal{X}$ such that $xEf(x)$ for every $x \in A$, the equality $m(f(A)) = m(A)$ holds.

The measure m is *quasi-invariant* with respect to the equivalence E if, for an arbitrary set A such that $m(A) = 0$, the union

$$[A]_E = \{x \in \mathcal{X}\colon\ xEa,\, a \in A\}$$

of the equivalence classes that intersect A also has the zero measure.

It is clear that, if the measure m is invariant (quasi-invariant) with respect to the action of the group G, then the measure m is invariant (quasi-invariant) with respect to the equivalence relation E_G. Since a uniform measure m on ∂T is invariant with respect to the action of any automorphism of the tree T, the equivalence E_G preserves the measure m for an arbitrary countable group G of automorphisms of the tree T.

Definition 6.7. A countable Borel equivalence $E \subseteq \mathcal{X} \times \mathcal{X}$ on the space with measure $(\mathcal{X}, m)$ is called *ergodic* if, for any measurable set A, the set $[A]_E$ has either the measure 1 or measure 0.

The equivalence E_G is ergodic if and only if the group G acts ergodically.

An important example of the equivalence relation defined by the action of a group on a rooted tree is given by the *confinality* relation.

Let $\mathbf{X} = \{X_1, X_2, \dots\}$ be a sequence of finite sets and $T = T(\mathbf{X})$ be a corresponding rooted tree.

For every $n \in \mathbb{N}$, we define the equivalence relation E_n on the boundary ∂T by the rule

$$(x_1x_2x_3\dots)E_n(y_1y_2y_3\dots) \quad \text{if and only if } x_i = y_i \text{ for all } i > n.$$

The relation E_n has finite equivalence classes.

For every $n \in \mathbb{N}$, the set of all automorphisms g of the tree T such that xE_nx^g for all x forms a finite group. This group is isomorphic to the group $\operatorname{Aut} T_n$ of automorphisms of a finite rooted tree T_n that consists of the vertices of the tree T lying on the levels with the numbers no greater than n. Obviously,

$$E_0 \subset E_1 \subset E_2 \subset \dots.$$

Denote $E_c = \bigcup_{n=0}^{\infty} E_n$. Two sequences $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \partial T(\mathbf{X})$ are in the relation E_c if and only if $x_i = y_i$ for all i except for a finite number. The relation E_c is called the *confinality* relation, and two sequences are called *confinal* if they are in the relation E_c. The equivalence classes of E_c are also called the *confinality classes*.

The following groups and semigroups of automorphisms of the tree T are related to the confinality relations E_n and E_c (see [47]).

Definition 6.8. The automorphism g of a tree T is called

(a) *finitary* or *rooted* if there exists $n \in \mathbb{N}$ such that, for any $x \in \partial T$, we have $x^g E_n x$;

(b) *weakly finitary* if, for any $x \in \partial T$, we have $x^g E_c x$;

(c) *confinal* if $E_c^g \subseteq E_c$, i.e., if, for arbitrary $x, y \in \partial T$ such that $x E_c y$, we have $x^g E_c y^g$.

It is obvious that the set of all finitary automorphisms of a tree forms a group. This group is the inductive limit of the groups $\operatorname{Aut} T_n$; hence, it is locally finite.

The set of all weakly finitary automorphisms also is a group, which is denoted by $\mathcal{AWF}(\mathbf{X})$. If T is a regular rooted tree, then the group $\mathcal{AWF}(\mathbf{X})$ contains an isomorphic copy of the whole group $\operatorname{Aut} T$ (see [47]).

The group $\mathcal{AWF}(\mathbf{X})$ is an analogue of *full groups* of minimal dynamical systems on a Cantor set that were investigated in [23].

The set of confinal automorphisms $\mathcal{AC}(\mathbf{X})$ is only a semigroup rather than a group since the inverse of a confinal automorphism is not necessarily confinal. As an example, we can consider a regular tree $T(\mathbf{X})$ constructed by the sequence $\mathbf{X} = \{\mathbb{Z}_2, \mathbb{Z}_2, \dots\}$ ($\mathbb{Z}_2$ is a group of order 2) and the automorphism

$$x_1 x_2 x_3 \dots \mapsto x_1(x_2 + x_1)(x_3 + x_2)(x_4 + x_3)\dots,$$

which is confinal; however, its inverse

$$x_1 x_2 x_3 \dots \mapsto x_1(x_1 + x_2)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4)\dots$$

is not confinal.

The semigroup $\mathcal{AC}(\mathbf{X})$ coincides with the semigroup

$$\{g \in \operatorname{Aut} T\colon\ \mathcal{AWF}(\mathbf{X})^g \le \mathcal{AWF}(\mathbf{X})\}.$$

In particular, $\mathcal{AWF}(\mathbf{X}) \le \mathcal{AC}(\mathbf{X})$, and the group $\mathcal{AC}(\mathbf{X}) \cap \mathcal{AC}(\mathbf{X})^{-1}$ (called a group of *biconfinal* automorphisms) coincides with the normalizer of the group $\mathcal{AWF}(\mathbf{X})$ in $\operatorname{Aut} T$.

The equivalence E_c is, in a sense, the minimal equivalence among ergodic equivalences defined by the actions of groups on the boundary of a tree. More precisely, the following proposition holds.

Proposition 6.23. *Let G be a group acting spherically transitively on a rooted tree $T(\mathbf{X})$ and E_G be the corresponding equivalence relation. Then, there exists an automorphism $g \in \operatorname{Aut} T$ such that $E_G^g \supseteq E_c$.*

Proof. Note that the condition $E_G^g \supseteq E_c$ is equivalent to the fact that the orbits of the group G^g are the unions of confinality classes.

Consider a certain infinite path $\{v_0, v_1, \dots\}$ without repetitions with the beginning in the root. Let G_n be a stabilizer of the vertex v_n in the group G. Note that $G = G_0$ and, for every $n \ge 1$, the group G_n is a subgroup of index m_n of the group G_{n-1}, where $\{m_1, m_2, \dots\}$ is the spherical index of the tree $T(\mathbf{X})$ (i.e., $m_n = |X_n|$).

Let us construct a *coset tree* whose vertices correspond to the right cosets of the group G with respect to the subgroups G_n and two vertices are incident if and only if one of the vertices corresponds to the coset $G_n \cdot g$ while the other corresponds to $G_{n+1} \cdot h$ such that $G_n \cdot g \supset G_{n+1} \cdot h$. As the root, we take the vertex corresponding to the class $G_0 \cdot 1$. On the cosets of G with respect to G_n, the group G naturally acts by right multiplication. The appropriate action on the coset tree is the action by automorphisms.

One can readily prove that the described action of the group G on the coset tree is conjugate to the original action of G on $T(\mathbf{X})$.

For every $n \geq 1$, choose a certain system of representatives H_n of right cosets of the group G_{n-1} with respect to the subgroup G_n. Since $|H_n| = |X_n|$, there exists a certain one-to-one correspondence $\psi_n \colon X_n \to H_n$.

The set of all possible products of the form $h_n \dots h_2 h_1$, where $h_i \in H_i$, is a system of representatives of the right cosets of the group G with respect to the subgroup G_n. Define a new action $\varphi \colon G \hookrightarrow \operatorname{Aut} T(\mathbf{X})$ of the group G on $\mathbf{X}^*$ so that the equality

$$(x_1 x_2 \dots x_n)^{\varphi(g)} = y_1 y_2 \dots y_n$$

is equivalent to the equality

$$G_n \psi_n(x_n) \dots \psi_2(x_2) \psi_1(x_1) g = G_n \psi_n(y_n) \dots \psi_2(y_2) \psi_1(y_1)$$

for any $x_i \in X_i$. This uniquely defines the action of the group G by automorphisms on the rooted tree $T(\mathbf{X})$, that is conjugate to the action of G on the coset tree and, hence, to the original action on the tree $T(\mathbf{X})$. Now, it suffices to prove that the orbits of the new action of G on $\partial T(\mathbf{X})$ are the unions of confinality classes.

Let $w_1 = a_1 a_2 \dots a_n x_{n+1} x_{n+2} \dots$ and $w_2 = b_1 b_2 \dots b_n x_{n+1} x_{n+2} \dots$ be two arbitrary confinal sequences from $\mathbf{X}^\omega$. Then, for

$$g = \left(\psi_n(a_n) \psi_{n-1}(a_{n-1}) \dots \psi_1(a_1)\right)^{-1} \psi_n(b_n) \psi_{n-1}(b_{n-1}) \dots \psi_1(b_1),$$

we have $w_1^{\varphi(g)} = w_2$; i.e., w_1 and w_2 lie on the same orbit. $\square$

Not any countable F_σ-equivalence containing E_{c} is the equivalence induced by the action of a countable automorphism group of a tree. The "tail" relation

$$(x_1, x_2, \dots) E_{\mathrm{t}} (y_1, y_2, \dots) \quad \Leftrightarrow \quad \exists k \in \mathbb{Z} \;\; \exists n \in \mathbb{N} \;\; \forall i > n \colon \; x_i = y_{i+k}$$

on the boundary of a regular rooted tree constructed by a constant sequence $\mathbf{X} = \{X, X, X, \dots\}$ may serve as such an example. Two sequences are in the relation E_{t} if and only if they differ only in the beginnings and not necessarily synchronously; i.e., the lengths of these beginnings may be different. The relation E_{t} does not preserve the measure on the boundary; therefore, it cannot be induced by the action of a countable automorphism group of the tree.

An example of the group that acts by homeomorphisms on $\mathbf{X}^\omega$ and whose orbits coincide with the equivalence classes of E_{t} is given by the Thompson group V. Indeed, any two sequences $w_1, w_2 \in \mathbf{X}^\omega$ such that $w_1 E_{\mathrm{t}} w_2$ can be represented as $w_1 = v_1 u$ and $w_2 = v_2 u$; then, the relation $w_1^g = w_2$ holds for any element g, of the group V, defined by the tableau of the form $\binom{v_1 \; \dots}{v_2 \; \dots}$.

Definition 6.9. An equivalence relation is called *finite* if all its classes are finite.

A Borel equivalence relation is called *hyperfinite* if it is a union of an ascending chain of finite Borel equivalences.

In the case of equivalence on a space with a quasi-invariant measure, the equivalence is usually called (hyper)finite if the conditions of Definition 6.9 are fulfilled almost everywhere.

The following characterization of hyperfinite equivalences is valid (see [13, 15]).

Theorem 6.24. *Let $\mathcal{X}$ be a Borel space. The Borel equivalence relation $E \subseteq \mathcal{X} \times \mathcal{X}$ is hyperfinite if and only if there exists a Borel action of the cyclic group $\mathbb{Z}$ such that $E = E_{\mathbb{Z}}$.*

Since the confinality is a union of the ascending chain of finite relations E_n, it is hyperfinite.

Another example of hyperfinite equivalence is the equivalence E_{t}, which, as we saw above, coincides with the equivalence induced by the action of the Thompson group V.

An example of the action on a tree of a cyclic group with the orbits that almost everywhere coincide with the confinality classes (except for one orbit that is a union of two confinality classes) is given by the action of the adding machine, which easily follows from its interpretation in terms of 2-adic numbers.

Proposition 6.25 [85]**.** *The orbits of action of the Grigorchuk group G on the boundary of the tree coincide with the confinality classes.*

Proof. Since each generator of the Grigorchuk group leaves every point of the boundary in its own confinality class, all orbits are subsets of confinality classes. It remains to prove that any two confinal sequences lie in the same G-orbit.

To this end, we prove by induction on n that, for arbitrary $w_1, w_2 \in X^*$ of the form

$$w_1 = a_1 a_2 \dots a_n x_{n+1} x_{n+2} \dots,$$
$$w_2 = b_1 b_2 \dots b_n x_{n+1} x_{n+2} \dots,$$

there exists an element $g \in G$ such that $w_1^g = w_2$. For $n = 1, 2$, the assertion is verified directly. If this assertion is true for $n-1$, then there exist $g_1, g_2 \in G$ such that

$$w_1^{g_1} = 11 \dots 10 a_n x_{n+1} x_{n+2} \dots,$$
$$w_2^{g_2} = 11 \dots 10 b_n x_{n+1} x_{n+2} \dots.$$

If $a_n = b_n$, then the assertion is true; otherwise, we have either $w_1^{g_1 b} = w_2^{g_2}$ or $w_1^{g_1 c} = w_2^{g_2}$. □

Using the conjugacy criterion for the automorphisms of a rooted tree from [67], we can prove the following theorem (see [47]).

Theorem 6.26. *Every element $g \in \operatorname{Aut} T$ is conjugate in the group $\operatorname{Aut} T$ to a certain element $h \in \mathcal{AWF}(\mathbf{X})$. Moreover, h can always be chosen so that it changes at most two letters in every sequence $x_1 x_2 x_3 \dots$.*

Corollary 6.27. *Every element of the automorphism group $\operatorname{Aut} T$ of a tree with the spherical index $(n, n, \dots, n)$ can be represented as a product of two elements of order at most $n!$.*

In particular, in the case of a binary tree, every element of the group $\operatorname{Aut} T$ is a product of two involutions.

Equivalence E is called *aperiodic* if it does not have any finite classes.

Two equivalences R_1 and R_2 on standard Borel spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ are called *Borel isomorphic* if there exists an isomorphism of Borel spaces $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ such that $x R_1 y$ if and only if $f(x) R_2 f(y)$.

A measure m on $\mathcal{X}_i$ is called *ergodic with respect to the equivalence R_i* if the equivalence R_i is ergodic on the space $(\mathcal{X}_i, m)$.

Let E_{t} be the above-defined "tail" equivalence on the set $\mathbf{X}^\omega$ for $\mathbf{X} = \{X, X, X, \dots\}$, $|X| = 2$.

Define the relation $E_c \times \Delta_n$ on the direct product $\mathbf{X}^\omega \times A$ of the space of sequences $\mathbf{X}^\omega$ and a set of n elements ($1 \le n \le \aleph_0$) by the condition

$$(x,a)(E_c \times \Delta_n)(y,b) \quad \Leftrightarrow \quad xE_c y, \quad a=b,$$

where E_c is the confinal equivalence relation. Note that $E_c \times \Delta_1$ is naturally identified with E_c.

One can easily verify that E_t has no invariant probability measures, while $E_c \times \Delta_n$ has exactly n of them.

Denote by E_s the equivalence induced by the action of the Bernoulli shift over a two-element alphabet that is restricted to the set of nonperiodic points. The equivalence E_s has a continuum of invariant ergodic probability measures, for example, the Bernoulli measures.

Dougherty, Jackson, and Kechris [15] classified the hyperfinite aperiodic equivalences up to the Borel isomorphism.

Theorem 6.28. *Two hyperfinite aperiodic equivalences are Borel isomorphic if and only if their sets of ergodic invariant probability measures are equipotent.*

The cardinality of the set of ergodic invariant probability measures of hyperfinite equivalence is either at most countable or continual.

Corollary 6.29. *Any hyperfinite aperiodic equivalence is Borel isomorphic to one of the following equivalences*:

$$E_t, \qquad E_c \times \Delta_n, \quad 1 \le n \le \aleph_0, \qquad E_s.$$

Corollary 6.30. *Any hyperfinite aperiodic equivalence that has at least one invariant probability measure is Borel isomorphic to a certain equivalence induced by the action of a finitely automatic automorphism of a binary tree on the boundary.*

Proof. If an equivalence has an invariant probability measure, then it is not isomorphic to the equivalence E_t, and its set of invariant ergodic probability measures is nonempty.

By Proposition 6.8, for any tree automorphism, the set of ergodic invariant probability measures and the boundary of the orbit tree are equipotent. The equivalence E_G is aperiodic if the group G has no finite orbits on the boundary. Therefore, to prove the corollary, it suffices to construct, for any $1 \le n \le \aleph_0$ and n equal to the cardinality of continuum, an automorphism of a binary tree that has exactly n different closures of orbits on the boundary, all of them being infinite.

For $n = 1$, the adding machine serves as such an automorphism. Other automata are also constructed with the use of the adding machine.

For a finite $n > 1$, this is the automorphism defined by the automaton depicted in Fig. 32a. Figure 32b illustrates the tree of orbits of the automorphism.

For $n = \aleph_0$, one can take the automaton depicted in Fig. 33.

When n is equal to the cardinality of a continuum, an example is given by the automaton depicted in Fig. 34a (again, Fig. 34b shows the corresponding tree of orbits).

The image of the sequence $x_1x_2x_3\ldots \in \mathbf{X}^\omega$ under the action of this last automaton is equal to $x_1y_2x_3y_4x_5y_6\ldots$, where $y_2y_4y_6\ldots$ is the image of the sequence $x_2x_4x_6\ldots$ under the action of the adding machine. $\square$

The concept of hyperfiniteness is closely related to the concept of amenability of the equivalence relation. In particular, for any amenable group G, the relation E_G is hyperfinite *almost everywhere* [16, 50]. In addition, if a group G acts freely by measure-preserving transformations and the relation E_G is hyperfinite, then the group G is amenable [20].

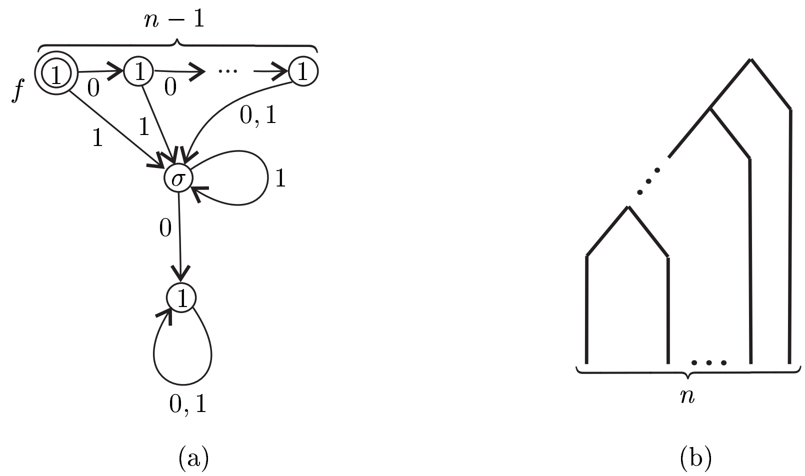


Fig. 32

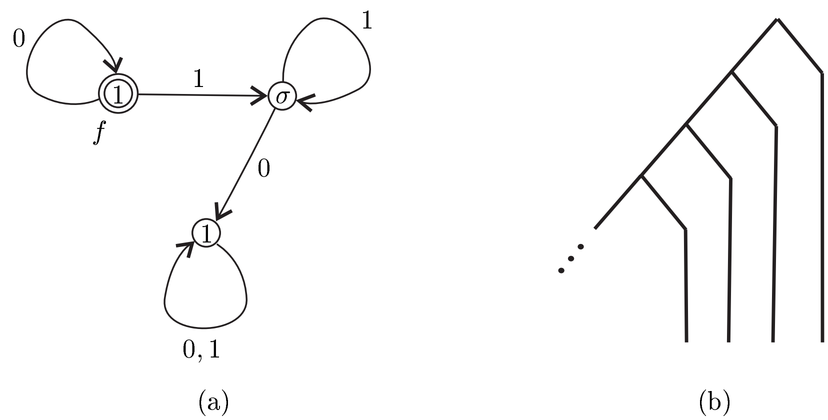


Fig. 33

As we noted above (Theorem 6.26), any cyclic group is conjugate to a subgroup of the group $\mathcal{AWF}(\mathbf{X})$. It is also easy to prove that any finite group is conjugate to a subgroup of the group $\mathcal{AWF}(\mathbf{X})$ (see [47, Theorem 6]). The proof of Theorem 6 from [47] is carried over virtually without changes to the proof of the following generalization of this theorem.

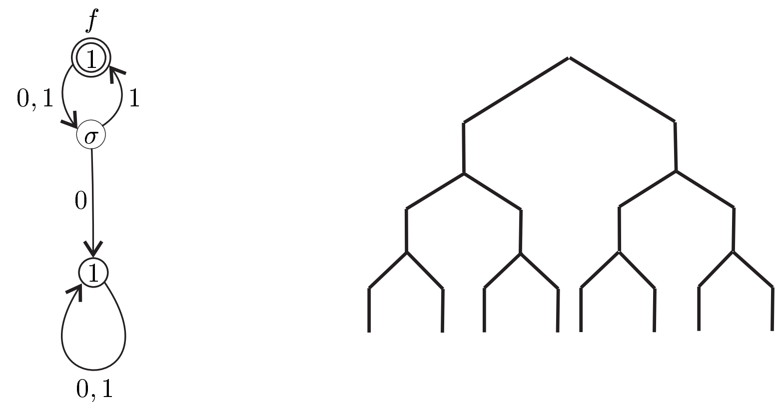
Theorem 6.31. *If, for a group $G \leq \operatorname{Aut} T$, all orbits on the boundary of the tree T are finite, then this group is conjugate in $\operatorname{Aut} T$ to a certain subgroup of the group $\mathcal{AWF}(\mathbf{X})$.*

Any Borel subequivalence of a hyperfinite equivalence is hyperfinite. Hence, the equivalence E_G for $G \leq \mathcal{AWF}(\mathbf{X})$ is hyperfinite. Is the converse true? To put it more precisely, the following question arises.

Question. Is it true that *an arbitrary countable group $G \leq \operatorname{Aut} T$ with the hyperfinite equivalence E_G is conjugate in $\operatorname{Aut} T$ to a subgroup of the group $\mathcal{AWF}(\mathbf{X})$*?

This question may serve as a step toward a full classification of hyperfinite (ergodic) equivalences induced by the actions on the boundary of countable groups of tree automorphisms up to the conjugacy in the group $\operatorname{Aut} T$.

There are examples of finitely generated groups of tree automorphisms such that the induced equivalences on the boundary are not hyperfinite, and, hence, the groups themselves cannot be



(a) (b)

Fig. 34

conjugate to the subgroups of the group $\mathcal{AWF}(\mathbf{X})$.

Indeed, let G be a finitely generated residually finite nonamenable group (for example, a free group). Suppose that $G = G_0 > G_1 > G_2 > \dots$ is the sequence of its normal subgroups of finite index with a trivial intersection. Construct a rooted tree whose vertices are the right cosets with respect to the subgroups G_n, where two vertices are adjacent if and only if they have the form $G_n a$, $G_{n+1} b$ and $G_n a > G_{n+1} b$. The group G acts on this tree by the right multiplication by automorphisms, the stabilizer of an arbitrary vertex of level n being equal to G_n; hence, the stabilizer of any end is trivial; therefore, the action on the boundary is free. The induced equivalence is not hyperfinite since G is a nonamenable group acting freely by measure-preserving transformations.

A wider class of groups that induce nonamenable equivalences can be obtained by the concept of a *subgraph of equivalence*.

A graph with the vertex set $\mathcal{X}$ is called Borel if the set of pairs of adjacent vertices $(x, y) \in \mathcal{X} \times \mathcal{X}$ is Borel.

Definition 6.10. Let E be a Borel equivalence on the space $\mathcal{X}$. A *subgraph of equivalence* E ("graphing") is an arbitrary Borel graph with the vertex set $\mathcal{X}$ in which two vertices may be linked by an edge only if they are in the relation E. A graph is called a *graph of bounded valence* if the valences of its vertices are bounded in the aggregate.

A subgraph Γ of equivalence E is called *generating* if its connected components coincide with the equivalence classes of E.

In the general case, the connected components of the subgraph of equivalence E are the subsets of equivalence classes.

If G is a group with a finite system of generators $S = S^{-1}$, then its Schreier graph $\Gamma(G, S, \mathcal{X})$ (i.e., a graph with the vertex set $\mathcal{X}$ in which two vertices x and y are adjacent if and only if $x^s = y$ for a certain $s \in S$) is a generating subgraph of the equivalence E_G. Thus, the concept of a generating subgraph is a generalization of the concept of the Schreier graph for Borel countable equivalences.

Definition 6.11. A graph Γ is *nonamenable* if there exist positive constants R and C such that, for an arbitrary finite set A of vertices of the graph Γ, the following inequality holds:

$$|\mathrm{Neib}(A, R)| \geq (1 + C)|A|,$$

where $\mathrm{Neib}(A,R)$ is the set of all vertices that are situated at a distance less than or equal to R from a vertex belonging to A (the R-neighborhood of the set A).

According to the Fölner criterion, a finitely generated group is amenable if and only if its Cayley graph is amenable.

The following theorem is valid (see [1, 34]).

Theorem 6.32. *Let E be a Borel equivalence on the measure space $(\mathcal{X},\mu)$. If E is hyperfinite and preserves the measure μ, then μ-almost all connected components of any subgraph of bounded valence of equivalence E are amenable.*

A subgraph of equivalence is called a *treeing* if all its connected components are trees.

An example of the equivalence for which there exists a generating treeing is the equivalence E_G, where G is a free group (in particular, a cyclic group) that acts freely. The corresponding treeing is given by the Schreier graph of the group action. Thus, for hyperfinite aperiodic equivalences, there always exists a generating treeing.

On the other hand, not any countable Borel equivalence has a generating treeing. Examples of ergodic group actions for which the orbital equivalence E_G does not have a generating treeing are, by virtue of the results of [2], the actions of groups with the Kazhdan T-property (see also [32]).

Paulin in [51] proved that almost all (in the sense of measure) components of the generating subgraph of a measurable countable equivalence have 0, 1, 2, or an infinite number of ends.

The orbital invariance of L^2-Betti numbers of measurable partitions was proved in [21].

7. LIST OF PROBLEMS

7.1. Growth of automata.

1. Give a classification of the types of growth of noninitial automata.

2. A similar problem for noninitial automata.

3. Calculate the growth asymptotics for concrete automata that have an intermediate growth, for example, the automata depicted in Figs. 8, 9, 14, and 15.

7.2. Groups and semigroups of automata.

1. Does there exist an algorithm that

(a) determines, by a given finite initial automaton, whether or not it is periodic, i.e., whether the equivalence $A_q^{(m+n)} = A_q^{(n)}$ holds for certain $m,n \in \mathbb{N}$?

(b) determines, by a given finite noninitial automaton A, whether the semigroup $S(A)$ (group $G(A)$) is finite, abelian, nilpotent, solvable, free, periodic, or of intermediate growth?

(c) determines, by a given finite noninitial automaton, whether it is an automaton of polynomial, exponential or intermediate growth?

(d) determines, by two automata A and B, whether or not the semigroups $S(A)$ and $S(B)$ are isomorphic? The same question is posed for the groups $G(A)$ and $G(B)$ in the case of invertible automata.

(e) determines whether or not an initial automaton A_q is spherically transitive?

(f) determines whether or not a noninitial automaton A is spherically transitive, i.e., whether the semigroup $S(A)$ is spherically transitive? A similar question for the group $G(A)$ in the case of invertible automata.

(g) determines whether the group $G(A)$ is fractal, branch, weakly branch, or rigid? A group acting on a tree is called *rigid* if the rigid stabilizers of all vertices are finite.

2. Is the conjugacy problem in the groups of finite automata solvable?

3. Does there exist a nonamenable group defined by finite synchronous automata without a free subgroup with two generators? A similar question in the asynchronous case: a candidate for this group is the Thompson group F.

4. Give a classification of groups of finite automata up to the commensurability or to a coarser equivalence defined in [53] (by the commensurability of two groups we mean the situation when these groups have isomorphic subgroups of finite index).

5. Under what conditions on the automorphism group of a homogeneous tree the vertex stabilizer acts by finitely automatic transformations? In particular, is this true for $\mathrm{SL}(2,\mathbb{Z}[1/p])$, where p is a prime number?

Certain other problems of the theory of (synchronous) automatic transformation groups are presented in [46].

7.3. Problems concerning the Schreier graphs $\Gamma(G,S)$.

1. Does there exist an algorithm that

(a) determines, by a given invertible finite automaton A and recursively defined path $w \in \partial T$, whether or not the parabolic subgroup $P_w = \mathrm{St}_G\, w$ is trivial?

(a′) determines if there exists a path w such that $P_w = 1$?

(b) determines whether the Schreier graph $G(A)/P_w$, where w is a recursively defined sequence, has a polynomial growth?

2. Describe all possible types of growth for the graphs $G(A)/P_w$, where A is a finite invertible automaton and P_w is a parabolic subgroup.

3. Does there exist a spherically transitive group of automorphisms of a rooted tree such that the Schreier graphs of orbits, on the boundary, which are typical in the sense of the Baire category (the orbits of generic points) are different from (not locally isomorphic to) the Schreier graphs of orbits which are typical in the sense of measure?

7.4. Problems of the spectral theory of automata.

1. To each automaton, there correspond two spectra: the spectrum of the dynamical system defined by this automaton on the boundary and the spectrum of the Schreier graph. Construct an example of the automaton for which these spectra do not coincide.

2. Give a classification for the topological types of spectra of finite automata.

3. Find new (as compared with those described in [7, 30]) computation methods for the spectra of finite automata.

7.5. Dynamical systems.

1. Give a topological and metric classification of rational homeomorphisms.

2. For a fixed spherically transitive group $G < \mathrm{Aut}\, T$, classify its actions on the boundaries that are induced by the spherically transitive actions of this group on rooted trees.

3. Does there exist an invertible automaton A such that the group $G(A)$ possesses the Kazhdan property? Does there exist an invertible automaton A such that the group $G(A)$ possesses the Kazhdan property and acts ergodically on the boundary of a tree?

4. Construct an invertible automaton A such that the group $G(A)$ is spherically transitive and its action on the boundary defines a nonamenable partition into orbits.

5. Give a classification of orbital equivalences E_G of the action, on the boundary, of the groups of tree automorphisms up to

(a) Borel isomorphisms of the boundary of the tree,

(b) homeomorphisms of the boundary of the tree,

(c) isometries of the boundary of the tree.

6. Give a classification of hyperfinite orbital equivalences E_G of actions on the boundary of the groups of tree automorphisms up to the isometries of the boundary of a tree.

7. Is it true that an arbitrary countable group $G \leq \operatorname{Aut} T$ with the hyperfinite equivalence E_G is conjugate in $\operatorname{Aut} T$ to a subgroup of the group $\mathcal{AWF}(\mathbf{X})$?

8. Does there exist a countable (finitely generated?) amenable subgroup of $\operatorname{Aut} T$ that is not conjugate to a subgroup of the group $\mathcal{AWF}(\mathbf{X})$?

9. Does there exist a cyclic automorphism group of a rooted tree whose orbits on the boundary coincide with the confinality classes?

10. Develop a method for calculating the L^2-Betti numbers for the partitions of a boundary for the actions of the groups $G(A)$, where A is a finite automaton.

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